GUDLAVALLERU ENGINEERING COLLEGE

(An Autonomous Institute with Permanent Affiliation to JNTUK, Kakinada)
Seshadri Rao Knowledge Village, Gudlavalleru – 521 356.

Department of Civil Engineering

HANDOUT

on

LINEAR ALGEBRA & DIFFERENTIAL EQUATIONS

Vision

To provide quality education embedded with knowledge, ethics and advanced skills and preparing students globally competitive to enrich the civil engineering research and practice.

Mission

- To aim at imparting integrated knowledge in basic and applied areas of civil engineering to cater the needs of industry, profession and the society at large.

- To develop faculty and infrastructure making the department a centre of excellence providing knowledge base with ethical values and transforming innovative and extension services to the community and nation.

- To make the department a collaborative hub with leading industries and organizations, promote research and development and combat the challenging problems in civil engineering which leads for sustenance of its excellence.

Program Educational Objectives

PEOI : Exhibit their competence in solving civil engineering problems in practice, be employed in industries and undergo higher studies.

PEOII : Adapt to changing technologies with societal relevance for sustainable development in the field of their profession.

PEO III: Develop multidisciplinary team work with ethical attitude &social responsibility and engage in life - long learning to promote research and development in the profession.

LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS

Class & Sem. :I B.Tech – I Semester Year: 2019-20
Branch : CE Credits : 4

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1. **Brief History and Scope of the Subject**

   "MATHEMATICS IS THE MOTHER OF ALL SCIENCES", It is a necessary avenue to scientific knowledge , which opens new vistas of mental activity. A sound knowledge of engineering mathematics is essential for the Modern Engineering student to reach new heights in life. So students need appropriate concepts, which will drive them in attaining goals.

   Scope of mathematics in engineering study :

   Mathematics has become more and more important to engineering Science and it is easy to conjecture that this trend will also continue in the future. In fact solving the problems in modern Engineering and Experimental work has become complicated, time – consuming and expensive. Here mathematics offers aid in planning construction, in evaluating experimental data and in reducing the work and cost of finding solutions.

   The most important objective and purpose in Engineering Mathematics is that the students becomes familiar with Mathematical thinking and recognize the guiding principles and ideas "Behind the science" which are more important than formal manipulations. The student should soon convince himself of the necessity for applying mathematical procedures to engineering problems.

2. **Pre-Requisites**

   Basic Knowledge of Mathematics such as differentiation and Integration at Intermediate Level is necessary.

3. **Course Objectives:**
   - To know different procedures to solve the system of linear equations.
   - To find the Eigenvalues and Eigenvectors.
   - To find the solutions of 1^{st} and 2^{nd} order Differential equations.

4. **Course Outcomes:**

   Students will be able to

   CO1: solve the system of linear equations by different methods.

   CO2: use the concepts of Eigenvalues and Eigenvectors in Engineering problems.

   CO3: apply 1^{st} and 2^{nd} order differential equations to various Engineering Problems.

   CO4: apply the techniques of partial differentiation to find maxima and minima of functions in two or three variables.

5. **Program Outcomes:**

   Graduates of the Civil Engineering Program will have

   1. **Engineering knowledge:** Apply the knowledge of mathematics, science, engineering fundamentals and an engineering specialization for the solution of complex engineering problems.

2. **Problem analysis:** Identify, formulate, research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design/development of solutions:** Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for public health and safety, and cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems:** Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage:** Create, select, and apply appropriate techniques, resources, and Modern engineering and IT tools, including prediction and modeling to complex engineering activities, with an understanding of the limitations.
6. **The engineer and society:** Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal, and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability:** Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics:** Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and team work:** Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication:** Communicate effectively on complex engineering activities with the engineering community and with the society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance:** Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.
12. **Life-long learning:** Recognizes the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

6. **Mapping of Course Outcomes with Program Outcomes:**

	1	2	3	4	5	6	7	8	9	10	11	12
CO1	3	2										
CO2	2	2										
CO3	2											
CO4	3	1										

7. **Prescribed Text Books**

   1. B.S.Grewal, Higher Engineering Mathematics : 42^{nd} edition, Khanna Publishers,2012 , New Delhi.

2. B.V Ramana, Higher Engineering Mathematics, Tata-Mc Graw Hill Company Ltd.

8. **Reference Text Books**
    1. U.M.Swamy, A Text Book of Engineering Mathematics – I & II : 2nd Edition, Excel Books, 2011, New Delhi.
    2. Erwin Kreyszig, Advanced Engineering Mathematics : 8th edition, Maitrey Printech Pvt. Ltd, 2009, Noida.

9. **URLs and Other E-Learning Resources**

   Sonet CDs & IIT CDs on some of the topics are available in the digital library.

10. **Digital Learning Materials:**
    - http://nptel.ac.in/courses/106106094
    - http://nptel.ac.in/courses/106106094/40
    - http://nptel.ac.in/courses/106106094/30
    - http://nptel.ac.in/courses/106106094/32
    - http://textofvideo. nptl.iitm.ac.in/106106094/lecl.pdf

11. **Lecture Schedule / Lesson Plan**

Topic	**No. of Periods**	
	Theory	**Tutorial**
UNIT –1: System of linear equations		
Rank of a matrix	**1**	
Echelon form	**1**	**2**
Normal form	**3**	
System of equations-Consistence and inconsistence	**2**	
Solving non_homogeneous system-LU Decomposition	**3**	**2**
UNIT-II : EIGEN VALUES AND EIGEN VECTORS		
Eigen values and Eigen vectors	**2**	**2**
Properties of eigen values and eigen vectors	**2**	
Cayley-Hamilton theorem	**2**	**2**
Finding inverse and power of a matrix	**2**	
UNIT-III: First order differential equations		
Exact D.E	**2**	
Non-exact D.E	**4**	**2**
Applications:Newtons law of cooling	**2**	**2**
Orthogonal trajectory	**2**	
UNIT-IV: Higher order linear ordinarydifferential equations		
Solving homogeneous D.E	**2**	
Finding Particular integral of Non-Homogenous D.E. when RHS is e^{ax}	**2**	**2**
Finding Particular integral of Non-Homogenous D.E. when RHS is Sin ax or Cos ax.	**2**	
Finding Particular integral of Non-Homogenous D.E. when RHS is a polynomial in x.	**2**	**2**
Finding Particular integral of Non-Homogenous	**2**	

D.E. when RHS is e^{ax}.(a function of x)		
Finding Particular integral of Non-Homogenous D.E. when RHS is x.(a function of x)	**2**	
UNIT-V:Partial differentiation		
Total derivative	**1**	**2**
Chain rule	**1**	
Maxima and Minima of functions of 2 or 3 variables with constraints	**3**	**2**
Maxima and Minima of functions of 2 or 3 variables without constraints	**3**	
UNIT-VI:First order P.D.E		
Forming P.D.E BY eliminating arbitrary functions	**2**	**2**
Lagranges linear equation	**3**	
Non-linear P.D.E- By Charpit's Method	**3**	**2**
Total No. of Periods:	**56**	**24**

12. **Seminar Topics**
- Formation of ODE in the case of falling a stone from a height h
- Modeling and solving higher order ODE for Electrical Circuits
- Finding Maxima volume of an object inscribed in another object

LINEAR ALGEBRA & DIFFERENTIAL EQUATIONS
UNIT-I
SYSTEM OF LINEAR EQUATIONS

Objectives:

- To introduce the concept of rank of a matrix.
- To know methods of solving system of Linear equations.
- To be familiar with LU-Decomposition method.

Syllabus:

Rank of a matrix-Echelon form, Normal form, system of equations-Consistence and inconsistence, solving non-homogeneous system of equations by LU-Decomposition.

Learning Outcomes:

Students will be able to

- Calculate rank of a matrix.
- Solve system of Linear equations using by LU-Decomposition
- find an LU decomposition of simple matrices and apply it to solve systems of equations , be aware of when an LU decomposition is unavailable and when it is possible to circumvent the problem

UNIT – I
LEARNING MATERIAL

Introduction of Matrices:
Definition :

A rectangular arrangement of mn numbers, in m rows and n columns and enclosed within a bracket is called a matrix. We shall denote matrices by capital letters as *A*,*B*, *C* etc

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{m \times n}$$

Remark: A matrix is not just a collection of elements but every element has assigned a definite position in a particular row and column.

Sub – Matrix: Any matrix obtained by deleting some rows or columns or both of a given
matrix is called sub matrix.

Minor of a Matrix: let A be an mxn matrix. The determinant of a square sub matrix of A is called a minor of the matrix.

Note: If the order of the square sub matrix is 't' then its determinant is called a minor of order 't'.

Rank of a Matrix:

Definition:

A matrix is said to be of rank r if

i. It has at least one non-zero minor of order r and
ii. Every minor of order higher than r vanishes.

Rank of a matrix A is denoted by ρ(A).

Properties:

1) The rank of a null matrix is zero.
2) For a non-zero matrix A , $\rho(A) \geq 1$
3) The rank of every non-singular matrix of order n is n. The rank of a singular matrix of order n is < n.
4) The rank of a unit matrix of order n is n.
5) The rank of an m × n matrix ≤ min(m ,n).
6) The rank of a matrix every element of which is unity is one
7) Equivalent matrices have the same order and same rank because elementary transformation do not alter its order and rank.
8) Rank of a matrix is unique.
9) Every matrix will have a rank

Elementary Transformations on a Matrix:

i) Interchange of i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$
ii) If i^{th} row is multiplied with k then it is denoted by $R_i \rightarrow k R_i$
iii) If all the elements of i^{th} row are multiplied with k and added to the corresponding elements of j^{th} row then if is denoted by $R_i \rightarrow R_i + kR_j$.

Note: 1. The corresponding column transformations will be denoted by writing 'c'

$$i.e\ c_i \leftrightarrow c_j,\ \ c_i \leftrightarrow k\,c_j\ ,\ c_i \rightarrow c_i + kc_j$$

2. The elementary operations on a matrix do not change its rank.

Equivalence of Matrices: If B is obtained from A after a finite chain of elementary transformations then B is said to be equivalent to A. It is denoted as B~A.

Different methods to find the rank of a matrix:

Method 1:

Echelon form: A matrix is said to be in Echelon form if

1) Zero rows, if any, are below any non-zero row
2) The number of zeros before the first non-zero elements in a row is less than the number of such zeros in the next rows.

Ex: The rank of matrix which is in Echelon form $\begin{bmatrix} 0 & 1 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 9 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is 3 since the no. of non-zero rows is 3

Note:1. Apply only row operations while reducing the matrix to echelon form

Problem: Find the rank of the matrix A = $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ by reducing into echelon form

Sol: A = $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

R_3-3R_1, R_4-R_1

$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$

R_3-R_2, R_4-R_2

$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The above matrix is in echelon form

Rank = no. of non zero rows = 2

Method 2:

<u>Normal Form</u>: Every m×n matrix of rank r can be reduced to the form of I_r, $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, $[I_r \quad 0]$, $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ by a finite chain of elementary row or column operations where I_r is the Identity matrix of matrix of order r.

Normal form another name is “canonical form”

Problem: Find the rank of matrix A = $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ by reducing it to canonical form

Sol: Given matrix A = $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ $R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 4 & 2 & 0 & 2 \\ 2 & -2 & 0 & 6 \\ 1 & -2 & 1 & 2 \end{bmatrix} R_2-4R_1, R_3-2R_1, R_4-R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix} C_2+C_1, C_4-3C$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix} R_2 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \end{bmatrix} (-1)R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \end{bmatrix} R_4-6R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -16 \end{bmatrix} C_3+C_2, C_4-C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -16 \end{bmatrix} \frac{1}{6}C_3$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&1&-16\end{bmatrix} C_4+16C_3$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&1&0\end{bmatrix} R_3 \leftrightarrow R_4$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix} \sim\begin{bmatrix}I_3&O\\O&O\end{bmatrix}$$

The above matrix is in normal form and rank is 3.

Elementary matrix:

A matrix obtained from a unit matrix by a single elementary transformation.

Ex: $\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$

Linear Equation: An Equation is of the form $a_1x_1+a_2x_2+a_3x_3+.....+....a_nx_n = b$ where $x_1,x_2,----,x_n$ are unknown and $a_1,a_2,...a_n$, b are constants is called a linear equation in 'n' unknowns.

Consistency of System of Linear equations (Homogeneous and Non Homogeneous) Using Rank of the Matrix:

A System of m linear algebraic equations in n unknowns $x_1,x_2,x_3,........x_n$ is a set of equations of the form

$$\left.\begin{array}{llll}a_{11}x_1 + & a_{12}x_2 + \ldots\ldots\ldots\ldots + & & a_{1n}x_n = b_1\\ a_{21}x_1 + & a_{22}x_2 + \ldots\ldots\ldots\ldots + & & a_{2n}x_n = b_2\\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & & \\ a_{m1}x_1 + & a_{m2}x_2 + \ldots\ldots\ldots\ldots +. & & a_{mn}x_n = b_m\end{array}\right\} \rightarrow (1)$$

The numbers a_{ij}'s are known as coefficients and b_i are known as constants of the system (1) can be expressed as $\sum_{j=i}^{n} a_{ij}x_j = b_i$,i = 1,2,....m

Non homogenous System **:** When at least one b_i is nonzero .

Homogenous System: If $b_i = 0$ for i = 1,2, ….m (all R.H.S constants are zero)
Solution of system (1) is set of numbers $x_1, x_2, x_3, \ldots\ldots x_n$ which satisfy simultaneously all the equations of the system (1)

Trivial Solution is a solution where all x_i are zero i.e $x_1 = x_2 = \ldots\ldots = x_n = 0$
The set of equations can be written in matrix form as AX = B → (2)

Where $A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots\ldots & & \\ a_{m1} & a_{m2} & \ldots\ldots\ldots & a_{mn} \end{bmatrix}$ is called the coefficient matrix

$X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$ is the set of unknowns $B = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix}$ is a column matrix of constants

<u>Consistent</u> : A system of equations is said to be consistent if (1) has at least one solution.
Inconsistent if system has no solution at all
Augmented matrix [A B] of system (1) is obtained by augmenting A by the column B
Matrix equation for the homogenous system of equations is AX = 0 ….(3)
It is always consistent.
If X_1, X_2 are two solutions of equation (3) then their linear combination $k_1x_1 + k_2x_2$ where k_1 & k_2 are any arbitrary numbers, is also solution of (3) .The no. of L.I solutions of m homogenous linear equations in n variables , AX = 0 , is (n – r) where r is the rank of the matrix A.

<u>Nature of solution:</u>
non-homogeneous with m equations and n unknowns
The system of equations AX=B is said to be

i. consistent and unique solution if rank of A = rank of [A|B] = n i.e.,r = n Where r is the rank of A and n is the no. of unknowns.
ii. Consistent and an infinite no. of solutions if rank of A= rank of [A|B] < n i.e., r < n. In this case we have to give arbitrary values to n-r variables and the remaining variables can be expressed in terms of these arbitrary values.
iii. Inconsistent if rank of A ≠ rank of [A|B]

Note: Method of finding the rank of A and [A|B]:

Reduce the augmented matrix [A:B] to Echelon form by elementary row transformations.

Problem: Show that the equations x+y+z = 4, 2x+5y-2z =3, x+7y-7z =5 are not consistent.

Sol: write given equations is of the form AX = B

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

Consider augmented matrix $\left[A/B\right] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{bmatrix}$

$$Applying\, R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{bmatrix}$$

$$Applying\, R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{bmatrix}$$

Thus $\rho(A) = 2 \; and \; \rho\left[A/B\right] = 3$

Therefore $\rho(A) \neq \rho\left[A/B\right]$

Hence the given system is an inconsistent.

Homogeneous linear equations:

Consider the system of m homogeneous equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{n2}x_2 + \dots\dots\dots + a_{mn}x_n &= 0 \end{aligned} \qquad (1)$$

(1) can be written as AX=O

Where A is the coefficient matrix formed by $A = \begin{bmatrix} a_{11} & a_{12} & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & a_{2n} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m2} & - & - & a_{mn} \end{bmatrix}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}$

Consistency: The matrix A and [A|B] are same. So rank of A = rank of [A|B]
Therefore the system (1) is always consistent.

Nature of solution:

Trivial solution: Obviously $x_1=x_2=x_3=$ -------- $=x_n=0$ is always a solution of the given
system and this solution is called trivial solution.
Therefore trivial solution or zero solution always exists.

Non-Trivial solution: Let r be the rank of the matrix A and n be the no. of unknowns.

Case-I: If r=n, the equations AX=O will have n-n i.e., no linearly independent solutions. In this case, the zero solution will be the only solution.

Case-II: If r<n, we shall have n-r linearly independent solutions. Any linear combination of
these n-r solutions will also be a solution of AX=O.

Case-III: If m<n then r≤m<n. Thus in this case n-r > 0.
Therefore when the no. of equations < No. of unknowns, the equations will have an infinite no. of solutions.

Note: The system Ax =0 possesses a non-zero solution if and only if A is a singular matrix.

Introduction of LU Decomposition :

A m×n matrix is said to have a **LU-decomposition** if there exists matrices L and U with the following
properties:

(i) L is a m×n lower triangular matrix with all diagonal entries being 1.
(ii) U is a m×n matrix in some echelon form.
(iii) $A = LU$.

Procedure to solve by LU Decomposition:
Suppose we want to solve a m×n system $A\boldsymbol{X} = \boldsymbol{b}$. If we can find a LU-decomposition for A , then to solve $A\boldsymbol{X} = \boldsymbol{b}$, it is enough to solve the systems $\left.\begin{matrix} LY = \boldsymbol{b} \\ UX = Y \end{matrix}\right\}$.

Thus the system LY = b can be solved by the method of forward substitution and the system $U\boldsymbol{X} = \boldsymbol{Y}$
can be solved by the method of backward substitution. To illustrate, we give some examples

It turns out that we need only consider lower triangular matrices L that have 1's down the diagonal. Here is an example, let A = $\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix}$ = LU where L = $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and U = $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ Multiplying out LU **and** setting the answer equal to A gives $\begin{bmatrix} u_{11} & u_{12} & u_{13} & \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} & l \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} & \end{bmatrix}$. Now we have to use this to find the entries in L and U. Fortunately this is not nearly as hard as it might at first seem. We begin by running along the top row to see that $u_{11} = 1$, $u_{12} = 5$, $u_{13} = 1$. Now consider the second row $l_{21}u_{11} = 2 \therefore l_{21} \times 1 = 2 \therefore l_{21} = 2$, $l_{21}u_{12} + u_{22} = 1 \therefore 2 \times 5 + u_{22} = 1 \therefore u_{22} = -9$, $l_{21}u_{13} + u_{23} = 3 \therefore 2 \times 1 + u_{23} = 3 \therefore u_{23} = 1$.
Now we solve the system LY=B i.e.,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 14/9 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 16 \end{bmatrix}$$ by forward substitution $y_1=9, y_2=-6, y_3=-5/3$

And the system UX=Y i.e., $\begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -5/9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ -5/3 \end{bmatrix}$ by backward substitution $x=1, y=1, z=3$.

UNIT-I

Assignment-cum-Tutorial Questions

A. Objective Questions

1. The rank of I_3 = __________

2. The rank of $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{pmatrix}$ is __________

3. If the rank of a matrix is 4. Then the rank of its transpose is_________

4. The rank of a matrix in echelon form is equal to _______

5. The necessary and sufficient condition that the system of equations AX=B is consistent if________________

6. The value of K for which the system of equations 5x+3y=12, 15x+9y=k-3 has infinitely many solution is ___________

7. The non trivial solution of system of equations $2x - 3y = 0$ *and* $-4x + 6y = 0$ *is* _____

8. The system of equations $x + y + z = 3, x + 2y + 3z = 4, x + 4y + 9z = 6$ will have _____

9. If the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 5 \\ 2 & k & 4 \end{bmatrix}$ is 2 then k=________

10. The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$

    (a) 0 (b) 1 (c) 2 (d) 3

11. If 5 non homogeneous equations are given with 4 unknowns. The system of equations AX=B consistent if

    (a) The rank of A=4 (b) the rank of A is 3

    (c) the rank of A <4 (d) the rank of A is 5

12. If the system of equations $x - 3y - 8z = 0, 3x + y - \lambda z = 0, 2x + 3y + 6z = 0$

    possess a nontrivial solution then λ =

    (a) 2 (b) $\frac{-4}{9}$ (c) 6 (d) 8

13. Every square matrix can be written as a product of lower and upper triangular matrices if

    (a)atleast one principal minor is zero (b) all principal minors are non-zero
    c) all principal minors are zero (d) atleast one principal minor is non-zero

14. Consider two statements:

    P: Every matrix has rank

    Q: Rank of a matrix is not unique

    (a) Both P and Q are false (b) Both P and Q are true

    (c) P is true and Q is false (d) P is false and Q is true

15. Which of the following statement is correct

    a. Rank of a Non-zero matrix is Zero

    b.Rank of a rectangular matrix of order mxn is m when m > n

    c.Rank of a rectangular matrix of order mxn is m when m < n

    d.Rank of a square matrix of order nxn is n+1.

    Rank of a non singular matrix of order m is

    a. m b. n c. 0 d. not defined

16. Rank of the matrix $A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}$ is

a. 1 b. 2 c. 3 d. 4

17. Find the values of k_1 and k_2 for which the non-homogeneous linear system, $3x - 2y + z = k_2$; $5x - 8y + 9z = 3$; $2x + y + k_1 z = -1$ has no solution

a) $k_1 = -3,\ k_2 = 1/3$ b) $k_1 = \quad 3,\ k_2 \neq 1/3$
c) $k_1 = -3,\ k_2 \neq 1/3$ d) $k_1 = \quad 3,\ k_2 = 1/3$

18. The equations $x + 4y + 8z = 16$, $3x + 2y + 4z = 12$ and $4x + y + 2z = 10$ have

a) only one solution b) two solutions
c) infinitely many solutions d) no solutions

B. Subjective Questions :

1. Determine the rank of matrix by reducing to echelon form

i) $A=\begin{bmatrix}1 & -1 & -1 & 2\\ 4 & 2 & 2 & -1\\ 2 & 2 & 0 & -2\end{bmatrix}$ ii) $A=\begin{bmatrix}3 & 2 & -1 & 5\\ 5 & 1 & 4 & -2\\ 1 & -4 & 11 & -19\end{bmatrix}$

iii) $A=\begin{bmatrix}-1 & -3 & 3 & -1\\ 1 & 1 & -1 & 0\\ 2 & -5 & 2 & -3\\ -1 & 1 & 0 & 1\end{bmatrix}$ $vi)$ $A=\begin{bmatrix}3 & -1 & 2 & 1\\ 1 & 4 & 6 & 1\\ 7 & -11 & -6 & 1\\ 7 & 2 & 12 & 3\end{bmatrix}$

2. Find the rank of the following matrices by reducing them into Normal form.

a) $\begin{bmatrix}1 & 2 & 3 & 0\\ 2 & 4 & 3 & 2\\ 3 & 2 & 10 & 3\\ 6 & 8 & 7 & 5\end{bmatrix}$ b) $\begin{bmatrix}1 & 2 & 3 & 2\\ 2 & 3 & 5 & 1\\ 1 & 3 & 4 & 5\end{bmatrix}$

3. Find the rank of the following matrices by reducing them into Canonical form

$\begin{bmatrix}1 & 3 & 4 & 5\\ 1 & 2 & 6 & 7\\ 1 & 5 & 0 & 10\end{bmatrix}$, $\begin{bmatrix}3 & -1 & 2\\ -6 & 2 & 4\\ -3 & 1 & 2\end{bmatrix}$

4. Test for the consistency and solve the following equations: $2x-3y+7z=5$; $3x+y-2z=13$; $2x+19y-47z=32$

5. Investigate for what values of a and b the simultaneous equations $x + ay + z = 3$; $x + 2y + 2z = b$; $x + 5y + 3z = 9$ have

a) no solution b) a unique solution c) infinitely many solutions

6. Test for consistency and solve if the equations are consistent $x+2y+2z=2, 3x-y+3z=-4, x+4y+6z=0$
7. Solve the system of equations by using LU Decomposition method
   3x+2y+2z=4, 2x+3y+z=5, 3x+4y+z=7.
8. Express A = $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$ as a product of LU. .
9. Test for the consistency of following and solve the following equations:
   x +2y +z =3 ; 2x +3y+2z =5; 3x -5y +5z =2; 3x +9y –z =4
10. For what value of k the equations x + y + z =1;2x + y + 4z = k;4x + y + 10z=k^2 have a solution and solve them completely in each case.

(C). GATE Previous Paper Questions

1. The system of linear equations $\begin{bmatrix} 2 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 5` \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}=\begin{bmatrix} 5 \\ -4 \\ 14 \end{bmatrix}$ has

   (GATE 2014)

   a) A unique solution b) infinitely many solutions
   b) No solution d) exactly two solutions
2. The system of equations x + y + z =6, x + 4y + 6z = 20, x + 4y + λz = u

   (GATE 2011)

   has no solution for values of λ and μ given by
   a) $\lambda=6, \mu=20$ b) $\lambda=6, \mu\neq 20$
   c) $\lambda\neq 6, \mu=20$ d) $\lambda\neq 6, \mu\neq 20$
3. The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is **(GATE 2006)**

   a) 0 b) 1 c) 2 d) 3
4. The determinant of a matrix A is 5 and the determinant of matrix B is 40 the determinant of matrix AB is_____________ **(GATE2014)**

5. Consider the following system of equations 3x + 2y = 1, 4x + 7z = 1, x + y + z =3, x- 2y+7z=0 The number of solution for this system is

   (GATE 2014)
6. The following system of equations $x_1+x_2+2x_3$ =1, x_1 + $2x_2+3x_3$ =2, $x_1+4x_2+\alpha x_3=4$ has α unique solution the only possible values of α are

   (GATE2008)

   a) 0 b) either 0 or 1 c) one of 0,1, or -1 d) any real number
7. Consider the following system of equations in three variables x_1,x_2 and x_3 $2x_1-x_2+3x_3=1$, $3x_1+2x_2+5x_3=2$, $-x_1+4x_2+x_3=3$ then The system of equations has **(GATE 2005)**

a) No Solutions b) More than one but a finite number of solutions
c) Unique solutions d) All infinite number of solutions

8. How many solutions does the following system of linear equations have –x

+ 5y = -1, x + 3y =3 , x – y =2 **(GATE 2013)**

a) Infinitely many b) Two distinct solutions

c) Unique d) None

9. For matrices of same dimension M and N and a scalar C which of these properties does not always hold **(GATE 2014)**

a) $(M^T)^T = M$ b) $(CM)^T = CM^T$

c) $(M+N)^T = M^T + N^T$ d) MN = NM

10. In the LU decomposition of the matrix $\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$, if the diagonal elements of U are both 1,then lower diagonal entry l_{22} of L is ___________.

(GATE 2009)

a) 4 b) 5 c) 6 d) 7

-----------------@-@-@-@-@---------------

LINEAR ALGEBRA & DIFFERENTIAL EQUATIONS

UNIT-II

EIGEN VALUES AND EIGEN VECTORS

Objectives:

- To understand eigen values, eigen vectors and their properties, cayley Hamilton theorem.

Syllabus:

Eigen values and eigen vectors, Properties of eigen values and eigen vectors (with out proof), Cayley-Hamilton theorem (with out proof)- Finding inverse and power of a matrix .

Course Outcomes:

Students will be able to

- Find eigen values and eigen vectors of a matrix
- Apply Cayley-Hamilton Theorem to compute powers and inverse of a given square matrix

UNIT – II

LEARNING MATERIAL

<u>Eigen values and eigen vectors of a matrix</u>:

Consider the following 'n' homogeneous equations in 'n' unknowns as given below

$$(a_{11} - \lambda)\, x_1 + a_{12}x_2 + \text{- - - - - -}\ a_{1n}x_n = 0$$

$$a_{21}x_1 + (a_{22} - \lambda)\, x2 + \text{- - - - - -}a_{2n}x_n = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n1}x_1 + a_{n2}x_2 + \text{- - - - - -}\ (a_{nn} - \lambda)x_n = 0$$

The above system of equations in matrix notation can be written as $(A - \lambda I)\, X = 0$

Where 'λ' is a parameter.

The matrix $(A - \lambda I)$ is called 'Characteristic Matrix' and $|A - \lambda I| = 0$ is called 'Characteristic Equation' of A . i.e.,

$|A - \lambda I| = (-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \text{- - - - - - -} + k_n = 0$

Where k_1, k_2,- - - - - ,k_n are expressible in terms of the elements a_{ij}

<u>Eigen Value:</u> The roots of characteristic equation are called the characteristic roots or latent roots or eigen values.

Eigen Vector: If λ is a characteristic root of a matrix then a non-zero vector X such that AX = λX is called a characteristic vector or Eigen vector of A corresponding to the characteristic root λ.

Note: (i) Eigen vector must be a non-zero vector

(ii) Eigen vector corresponding to a eigen value need not be unique

PROPERTIES OF THE EIGEN VALUES:

- The sum of the Eigen values of the square matrix is equal to its trace and product of the Eigen values is equal to its determinant.
- If λ is an eigen value of A corresponding to the eigen vector X then λ^n is the eigen value of the matrix A^n corresponding to the eigen vector X .
- If $\lambda_1, \lambda_2, \lambda_3$, ----, λ_n. are the latent roots of A then A^3 has the latent roots as $\lambda^3{}_1$, $\lambda^3{}_2$, $\lambda^3{}_3$, ----, $\lambda^3{}_n$.
- A square matrix A and its transpose A^T have the same eigen values.
- If A and B are n rowed square matrix and if A is invertible then $A^{-1}B$ and BA^{-1} have the same eigen values.
- If $\lambda_1, \lambda_2, \lambda_3$, ----, λ_n. are the eigen values of a matrix A then k λ_1, kλ_2,k λ_3, ----, kλ_n. are the eigen values of the matrix KA where K is a non-zero scalar.
- If λ is an Eigen value of the matrix A then λ+k is an Eigen value of the matrix A+KI.
- If λ is the Eigen value of A then λ-K are the eigen values of the matrix A-KI.
- If λ_1, λ_2 , λ_3 , ----, λ_n. are the eigen values of a matrix A then $(\lambda_1 - \lambda)^2$, $(\lambda_2 - \lambda)^2$,………… $(\lambda n - \lambda)^2$ are the eigern values of the matrix $(A- \lambda I)^2$.
- If λ is an Eigen value of a non-singular matrix A then λ^{-1} is an Eigen value of the matrix A^{-1} corresponding to the eigen vector X .
- If λ is an Eigen value of a non-singular matrix A then $|A|/\lambda$ is an eigen value of the matrix adjA.
- If λ is an Eigen value of a non-singular matrix A then $1/\lambda$ is an eigen value of A^{-1}.
- If λ is an Eigen value of a non-singular matrix A then the eigen value of B = $a_0A^2+a_1A+a_2I$ is $a_0\lambda^2+a_1\lambda+a_2$.
- The eigen values of a triangular matrix are just diagonal elements of the matrix.
- If A and B are non-singular matrices of same order,then AB and BA have the same eigen values.
- Suppose A and P are square matrices of order n such that P is non-singular,then A and $P^{-1}AP$ have the same eigen values.
- The eigen values of real symmetric matrix are real.

- For a real symmetric matrix ,the eigen vectors corresponding to two distinct eigen values are orthogonal.
- The two eigen vectors corresponding to two different eigen values are linearly independent.

Finding Eigen vectors:

Method1:

Case(i): Eigen values are distinct $\lambda_1 \neq \lambda_2 \neq \lambda_3$ (suppose the matrix A of order 3)

Corresponding to the eigen value λ_1, the eigen vector $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ can be obtained from the matrix equation $(A- \lambda_1)X_1=O$ and by expanding it we get three homogeneous linearly independent equations are obtained and solving any two equations for x_1,x_2,x_3 the eigen vector

$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ can be obtained. Similarly, the remaining Eigen vectors X_2, X_3 can be obtained corresponding to the Eigen values λ_2 and λ_3.

Case(ii): Finding Linearly Independent Eigen vectors of a matrix when the Eigen values of the matrix are repeated $(\lambda_1=\lambda_2)$

The matrix equation $(A- \lambda I)\ X=O$ gives three equations which represent a single independent equation of the form.

$a_1x_1+a_2x_2+a_3x_3 = 0$

We have to choose two unknowns as k_1,k_2.

So we can get two linearly independent Eigen vectors X_1 and X_2

Method2: (Rank method) in the matrix equation $(A-\lambda I)\ X=O$, reduce the coefficient matrix to Echelon form, the rank of the coefficient matrix is less than the number of unknowns. So give arbitrary constants to (n-r) variables and solve as in case of homogeneous equations.

Example 1: Find the Eigen values and corresponding Eigen vectors of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Sol: The characteristic equation of the matrix A is $|A- \lambda I| = 0 \Rightarrow$
$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^3 - 10\lambda^2 + 20\lambda - 32 = 0 \Rightarrow \quad (\lambda-2)(\lambda-2)(\lambda-8) = 0 \Rightarrow \quad \lambda = 2, 2, 8$$

The Eigen values of A are 2, 2 and 8.

Let the Eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A corresponding to the Eigen value λ is given by the non-zero

solution of the equation (A-λ I) X = O

$$\Rightarrow \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If λ=8, then the Eigen vector X_1 is given by (A-8I) X_1 = O

$$\Rightarrow \begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $-2x_1-2x_2+2x_3 = 0, -2x_1-5x_2-x_3 = 0, 2x_1-x_2-5x_3 = 0$

Solving any two of the equations, we get $\Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$ =k (say)

$\Rightarrow x_1=2k,\ x_2= -k,\ x_3=k$ (k is arbitrary) $\Rightarrow X_1= \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix} \Rightarrow X_1 = k\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

The eigen vector corresponding to λ_1=8 is $X_1= \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

If λ=2, then the Eigen vector X_2 is given by (A-2I) X_2 = O

$$\Rightarrow \begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $4x_1-2x_2+2x_3 = 0$, $-2x_1+x_2-x_3 = 0$, $2x_1-x_2+x_3 = 0$ $\Rightarrow$ $2x_1-x_2+x_3 = 0$

Let $x_2 = k_2$, $x_3 = k_1$, $2x_1 = k_2-k_1$ $\therefore X_2= \begin{bmatrix} \frac{k_2-k_1}{2} \\ k_2 \\ k_1 \end{bmatrix}$ =2 $\begin{bmatrix} \frac{k_2-k_1}{} \\ 2k_2 \\ 2\,k_1 \end{bmatrix}$ = $2k_1 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ +$2k_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\therefore$ Eigen vectors corresponding to $\lambda = 2$ are $X_2 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ $X_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\therefore$ Eigen vectors of A are $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Cayley-Hamilton theorem:

- *Every square matrix satisfies its own characteristic equation.*

Remark: (i) Determination of A^{-1} using Cayley-Hamilton theorem

Let A be n-rowed square matrix.By Cayley-Hamilton theorem ,A satisfies its own characteristic equation. i.e $(-1)^n[A^n+a_1A^{n-1}+a_2A^{n-2}+\dots\dots+a_nI] = 0$

$\Rightarrow A^n + a_1A^{n-1}+a_2A^{n-2}+\dots\dots+a_nI = 0$ ------ (1)

$\Rightarrow A^{-1}[\ A^n + a_1A^{n-1}+a_2A^{n-2}+\dots\dots+a_nI] = 0$

If A is a non-singular ,then we have $a_nA^{-1} = -A^{n-1}-a_1A^{n-2}-\dots\dots-a_{n-1}I$

$\Rightarrow A^{-1} = \left(\frac{-1}{a_n}\right)[A^{n-1}+a_1A^{n-2}-\dots\dots+a_{n-1}I]$

Remark: (ii) Determination of powers of **A** using Cayley-Hamilton theorem

Multiplying equation (1) with A, we get $A^{n+1} + a_1A^n+a_2A^{n-1}+\dots\dots+a_nA = 0$

$\Rightarrow A^{n+1} = -[a_1A^n+a_2A^{n-1}+\dots\dots+a_nA]$

Example 1: If A = $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ verify Cayley-Hamilton theorem .Find A^4 and A^{-1} using Cayley-Hamilton theorem**.**

Solution: Given matrix is $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

Characteristic equation is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & -1 \\ 2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(\lambda^2-2\lambda-3)-2(6-2\lambda)-1(-6+2\lambda) = 0$$

$\Rightarrow -\lambda^3 + 3\lambda^2 + 3\lambda - 9 = 0 \Rightarrow \lambda^3 - 3\lambda^2 - 3\lambda + 9 = 0$ ------(i)

$$A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{1} & \mathbf{-2} \\ \mathbf{2} & \mathbf{-2} & \mathbf{1} \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix}$$

Consider $A^3-3A^2-3A+9I = \begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix} -3 \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} -3 \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} +9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = O$

$A^3-3A^2-3A+9I = O$ -------- (ii)

Matrix A satisfies its own characteristic equation

Cayley-Hamilton theorem is verified by A

To find A^{-1} :Multiplying equation (ii) with A^{-1} on both sides

$$A^{-1}[\ A^3-3A^2-3A+9I] = A^{-1}(\ O) \Rightarrow A^2-3A-3I+9A^{-1} = 0$$

$$\Rightarrow 9A^{-1} = 3A+3I- A^2 = 3\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} +3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 6 & -3 & 0 \\ 6 & -6 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{9}\begin{bmatrix} 3 & 0 & 3 \\ 6 & -3 & 0 \\ 6 & -6 & 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

To find A^4 : Multiplying equation (ii) with A on both sides

$$A[\ A^3-3A^2-3A+9I] = A(\ O) \Rightarrow A^4-3A^3-3A^2+9A = O$$

$$\Rightarrow A^4 = 3A^3+3A^2-9A = 3\begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix} + 3\begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} -9 \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 72 & -72 \\ 0 & 81 & -72 \\ 0 & 0 & 9 \end{bmatrix}$$

UNIT-II
Assignment-cum-Tutorial Questions

A). Objective Questions

1. Two of the eigen values of a 3×3 matrix whose determinant equals 4 are -1 and 2 then the third eigen value of the matrix is equal to_______________________
2. The Eigen values of A= $\begin{bmatrix} 1 & 0 & -0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are ______________________________
3. If the Eigen values of A are 1,3,0 then $|A|$ = ________________
4. The Eigen values of A are (1,-1,2) then the eigen values of Adj(A) are __________________
5. If one of eigen values of A is 0 then A is ______________
6. The eigen value of adj A is ________________
7. If A is orthogonal then A^{-1}= ____________
8. Can an eigen vector be a zero vector?(yes/no)
9. The eigen values of A^2 are _____________where A= $\begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$
10. Can a zero value be an eigen value?(yes/no)
11. If 2,1,3 are the eigen values of A then the eigen values of B=3A+2I are ______________
12. If A is a singular matrix then _______________is an eigen value.
13. Identify the relation between geometric and algebraic multiplicity.
14. The sum of two eigen values and trace of a 3×3 matrix are equal then the value of $|A|$ is_____________________
15. Compute characteristic equation of A= $\begin{bmatrix} 3 & -2 & -8 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.
16. The matrix A has eigen values $\lambda_i \neq 0$ then A^{-1}-2I+A has eigen values ----

19.The Eigen values of A are 2,3,4 then the Eigen values of 3A are __ [ ]

(a)2,3,4 (b) $\frac{1}{2},\frac{1}{3},\frac{1}{4}$ (c) -2,3,2 (d) 6,9,12

20.If A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then A^3 = [ ]

(a) $2A^2+5A$ (b) $4A^2+2A$ (c) $2A^2+5A$ (d) $5A^2+2A$

B. Subjetive Questions :

1. Find the eigen values and eigen vectors of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$
2. Obtain the latent roots and latent vectors of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

3. Find the eigen values and eigen vectors of $\begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{bmatrix}$

4. Find the characteristic values and characteristic vectors of $\begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$

5. Verify that sum of eigen values is equalto trace of A for A= $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ and find the corresponding eigen vector.

6. Verify Cayley Hamilton theorem for A= $\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ Hence find A^{-1} and A^4

7. Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$. Hence find A^{-1} and A^4

8. For the matrix A = $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ find the eigen values of $3A^3+5A^2-6A+2I$.

9. For the matrix A= $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 2 & 5 \end{bmatrix}$ Find the eigen values and eigen vectors of A^{-1}

10. Using Cayley Hamilton theorem find A^4 for the matrix A= $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

(C). GATE Previous Paper Questions:

1. Eigen vector of the matrix $\begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$ is [ ]

(GATE-2004)

a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ b) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ d) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

2. For the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ the eigen value corresponding to eigen vector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$ is [ ]

(GATE-2006)

a) 2 b) 4 c) 6 d) 8

3. The eigen value of the matrix $\begin{bmatrix} 5 & 3 \\ 3 & -3 \end{bmatrix}$ is [ ]

(GATE-1999)

a)6 b)5 c) -3 d)-4

4. The 3 characteristic roots of A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ are [ ]

(GATE-2000)

a)2,3,3 b)1,2,2 c)1,0,0 d)0,2,3

5. The sum of the eigen values of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are [ ]

(GATE-2004)

a)5 b)7 c)9 d)18

6. Eigen values of S=$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ are 5 and 1.Eigen values of S^2=SS are[ ]

(GATE-2006)

a)1,25 b)6,4 c)5,1 d)2,10

7. One of the eigen vectors of A=$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ is [ ]

(GATE-2010)

a)$\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ b)$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ c)$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ d)$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

8. The minimum and maximum eigen value of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are -2, 6.what is other eigen value? [ ]

(GATE-2007)

a)5 b)3 c)1 d)-1

9. All the four entries of 2x2 matrix P=$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ are non-zero and one of its eigen value is zero which of the following is true? [ ]

(GATE-2008)

a)$p_{11}p_{22}-p_{12}p_{21}=1$ b)$p_{11}p_{22}-p_{12}p_{21}=-1$

c)$p_{11}p_{22}-p_{12}p_{21}=0$ d)$p_{11}p_{22}+p_{12}p_{21}=0$

10. Eigen values and the corresponding eigen vectors of a 2x2 matrix are given by

Eigen value	**Eigenvector**
$\lambda = 8$	$X=\begin{bmatrix}1\\1\end{bmatrix}$
$\mu = 4$	$Y=\begin{bmatrix}1\\-1\end{bmatrix}$

Then the matrix is [ ]

(GATE-2006)

a) $\begin{bmatrix}6 & 2\\2 & 6\end{bmatrix}$ b) $\begin{bmatrix}4 & 6\\6 & 4\end{bmatrix}$ c) $\begin{bmatrix}2 & 4\\4 & 2\end{bmatrix}$ d) $\begin{bmatrix}4 & 8\\8 & 4\end{bmatrix}$

11. The characteristic equation of A is $t^2-t-1=0$, then [ ]

(GATE-2000)

a) A^{-1} does not exist b) A^{-1} exist but cannot be determined from the data
c) $A^{-1}=A+I$ d) $A^{-1}=A-I$

12. A particular 3x3 matrix has an eigen value -1.The matrix A+I reduces to $\begin{bmatrix}1 & 0 & -2\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$,corresponding to eigen value -1 ,all eigen vectors of A are non-zero vectors of the form [ ]

(GATE-2002)

a) $\begin{bmatrix}2t\\0\\t\end{bmatrix}, t\in R$ b) $\begin{bmatrix}2t\\s\\t\end{bmatrix} s,t\in R$ c) $\begin{bmatrix}t\\0\\-2t\end{bmatrix} t\in R$ d) $\begin{bmatrix}t\\s\\2t\end{bmatrix} s,t\in R$

13. By Cayley-hamilton theorem A = $\begin{bmatrix}-3 & 2\\-1 & 0\end{bmatrix}$ satisfies the relation [ ]

(GATE-2007)

a)$A+3I+2A^2=0$ b)$A^2+2A+2I=0$ c) $(A+I)(A+2I)=0$ d)$\exp(A)=0$

14. From question (13), A^9 = [ ]

a)511A+510I b)309A+104I c)154A+155I d)exp(9A)

15. The number of linearly independent eigen vectors of $\begin{bmatrix}2 & 1\\0 & 2\end{bmatrix}$ is [ ]

(GATE-2007)

a) 0 b)1 c)2 d)infinite

Unit –III

FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

Objectives:

To introduce basic methods to solve 1^{st} order ODE and applications of 1^{st} order ODE such as Newton's law of cooling and orthogonal trajectories.

Syllabus:

Exact and non-exact D.E., Applications : Newton's law of cooling and orthogonal trajectories.

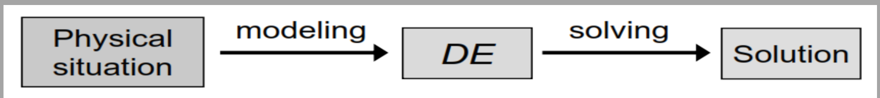
Outcomes:

At the end of the unit Students will be able to

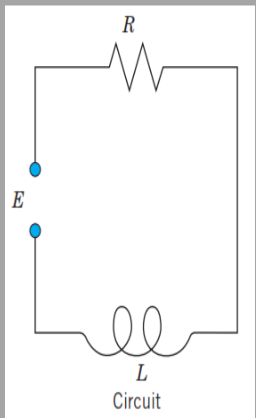
- differentiate exact and non-exact D.E
- solve exact and non-exact D.E
- apply the concept of Newton's law of cooling
- find orthogonal trajectory of given family of curves.

INTRODUCTION :

- If we want to solve an engineering problem (usually of a physical nature), we first have to formulate the problem as a mathematical expression in terms of variables, functions and equations. Such an expression is known as a ***mathematical model*** of the given problem.
- The process of setting up a model, solving it mathematically, and interpreting the result in physical or other terms is called mathematical modeling or, briefly, ***modeling.***
- Many physical concepts, such as velocity and acceleration, are derivatives. Hence a model is very often an equation containing derivatives of an unknown function. Such a model is called a ***differential equation.***
- Hence any Physical situation involving motion or measure rates of change can be described by a mathematical model, the model is just a differential equation.



Formation of Differential equations for real life problems →



Modeling RL-Circuit :
In this case, we use the following Physical Laws to create mathematical model.

[Ohm's law] → A current I in the circuit causes a **voltage drop RI** across the resistor

[Kirchoff's Voltage law] → A voltage drop $L\frac{dI}{dt}$ across the conductor, and the sum of these two voltage drops equals the EMF.

According to the above laws, the differential equation corresponding to the model is given by

$$L\frac{dI}{dt}+RI=E(t)$$

Differentiation :

❖ The rate of change of a variable w.r.t the other variable is called a differentiation.

In this case, changing variable is called *Dependent variable* and other variable is called an *Independent variable.*

Example **:** $\frac{dy}{dx}$ *is known as differentiation where y is dependent variable and x is independent variable.*

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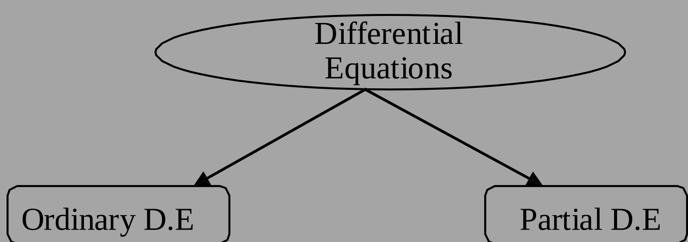
Differential Equations are separated into two types



➢ Ordinary D.E: In a D.E if there exist ***single Independent variable***, it is called as Ordinary D.E.

Example: 1) $\frac{dy}{dx}+2y=0$ is an Ordinary D.E 2) $x\frac{d^2y}{dx^2}+\frac{dy}{dx}+1=0$ is an Ordinary D.E.

➢ Partial D.E: In a D.E if there exist ***more than one Independent variables*** then it is called as Partial D.E

Example: 1) $\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=0$ is a Partial D.E.Here u depends on two independent variables x & y.

2) $\frac{\partial^2u}{\partial x\partial y}+1=0$ is a Partial D.E. Here u depends on two independent variables x & y.

****

- **Order of D.E. :**

  The order of the D.E. is the order of the highest derivative involving in the equation.

Example : **1)** Order of $\frac{d^2y}{dx^2}+2y=0$ is **Two.** **2)** Order of $\frac{d^5y}{dx^5}+\left[\frac{d^3y}{dx^3}\right]^8+3y=0$ is **Five**

- **Degree of D.E.:**

  The degree of the D.E is the degree of the highest ordered derivative involving in the equation, when the equation is free from radicals and fractional terms.

Example: 1) The degree of $\left[\frac{d^2y}{dx^2}\right]^1+2\frac{dy}{dx}+1=0$ is One.

2) The degree of $x\left[\frac{d^2y}{dx^2}\right]^8+\left[\frac{dy}{dx}\right]^{11}+\left[\frac{d^3y}{dx^3}\right]^2=0$ is Two.

ODE :

- Ordinary differential equation is an equation involving dependent variable (y) and its derivatives (y^1, y^{11},…) with respect to the independent variable (x).

Examples : $\frac{dy}{dx}+xy^2-4x^3=0$, $\frac{d^2y}{dx^2}+4\frac{dy}{dx}-3y=x^2-7$, …..

****

1st Order ODE :

- 1st Order Ordinary differential equation is an equation involving dependent variable (y) and its derivative y^1 with respect to the independent variable (x).

Examples : $\frac{dy}{dx}+xy^2-4x^3=0$

Solving 1st order & 1st degree ODE

We are going to solve the 1st order ODEs by the following methods.

1. Exact DE 2. Non-exact DE

Exact DE

- **Definition :** A D.E. which can be obtained by direct differentiation of some function of x and y is known as exact differential equation.

❖ Necessary & Sufficient condition for the D.E. of the form M(x, y) dx + N(x, y) dy = 0 to be ***Exact*** *is* $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

➢ **Procedure to solve Exact D.E.:**

Step 1 : Identify M and N

Step 2: Check of Exactness.

Step 3: If exact, General Solution is

$\int M \, dx + \int N \, dy = C$ **[In N, take terms which have no x variable]**

Problems :

1. Solve

$$\left\{ y\left(1+\frac{1}{x}\right) + \cos y \right\} dx + (x + \log x - x \sin y)\, dy = 0$$

<u>*Solution :*</u>

Step 1 : *Clearly* $M = y + \frac{y}{x} + Cos\, y$ & $N = x + \log x - x.Sin\, y$

$$\therefore \frac{\partial M}{\partial y} = \left(1+\frac{1}{x}\right) - \sin y \text{ and } \frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

Step 2: $\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, hence the given equation is exact

No "x" terms

Step 3: General Solution is given by $\int M \, dx + \int N \, dy = C$

$$\int \left\{ y\left(1+\frac{1}{x}\right) + \cos y \right\} dx + \int 0\, dy = C \Rightarrow y.(x + \log x) - Cosy.(x) = c$$

---

2. Solve (1 – sin x tan y) dx + (cos x sec^2 y) dy = 0

<u>*Solution :*</u>

Step 1 : Clearly , M = 1 – sin x tan y and N = cos x sec^2 y

No "x" terms

Step 2 : $\frac{\partial M}{\partial y} = -\sin x \sec^2 y = \frac{\partial N}{\partial x}$ $\Rightarrow$ Exact Differential

Step 3 : Hence General solution :

$$\int M\ dx + \int N\ dy = C$$

$$\Rightarrow \int (1 - Sinx.Tany)dx + 0dy = c$$

$$\Rightarrow x - (Tany).(-Cosx) = c \quad \text{or} \quad x + (Tany).(Cosx) = c$$

****

Non-Exact DE

❖ **If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the D.E. M(x, y) dx + N(x, y) dy = 0 is said to be Non-Exact Differential equation.**

❖ **Procedure to solve Non - Exact D.E.:**

> **Step 1 : Identify M and N**
>
> **Step 2: Check of Exactness.**
>
> **Step 3: If Non- Exact, <u>Convert the given D.E. to EXACT D.E</u> Using the Integrating Factor by the following suitable method.**

METHOD – 1 : Method to find Integrating factor $\frac{1}{Mx + Ny}$

> If given D.E. Mdx + Ndy = 0 is Non-Exact and ***<u>M, N are homogeneous functions of same degree</u>***, then I.F. $= \frac{1}{Mx + Ny}$

METHOD –2 : Method to find Integrating factor $\frac{1}{Mx - Ny}$

> If given D.E. Mdx + Ndy = 0 is Non-Exact and ***<u>M is of the form y.f(xy) & N is of the form x.g(xy)</u>****, then* I.F. $= \frac{1}{Mx - Ny}$

METHOD –3 : Method to find Integrating factor $e^{\int f(x)dx}$

> If given D.E. Mdx + Ndy = 0 is Non-Exact and
>
> if $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ = a function of x alone =f(x) then I.F. = $e^{\int f(x)dx}$

METHOD –4 : Method to find Integrating factor $e^{\int g(y)dy}$

If given D.E. Mdx + Ndy = 0 is Non-Exact and

if $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ = a function of y alone =g(y) then I.F. = $e^{\int g(y)dy}$

METHOD –5 : [Inspection Method] Observe the D.E. and if possible split the D.E. into any of the following R.H.S. and Integrate.

a. $d\left(\frac{x^2+y^2}{2}\right) = xdx + ydy$

b. $d(xy) = xdy + ydx$

c. $d\left(\frac{x}{y}\right) = \frac{ydx-xdy}{y^2}$

d. $d\left(\frac{e^y}{x}\right) = \frac{x.e^y dy - e^y dx}{x^2}$

e. $d\left(log\frac{y}{x}\right) = \frac{xdy-ydx}{xy}$

$d\left(log\frac{x}{y}\right) = \frac{ydx-xdy}{xy}$

g. $d\left(Tan^{-1}\frac{x}{y}\right) = \frac{ydx-xdy}{x^2+y^2}$

h. $d\left(Tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$

Example 1 :

1. Solve $x^2y\,dx - (x^3 + y^3)\,dy = 0$

∴ Non-Exact D.E

Solution :

Step 1 : Here $M = x^2y$ and $N = -x^3 - y^3$

Clearly, $\frac{\partial M}{\partial y} = -3y^2 \neq \frac{\partial N}{\partial x} = 2xy$.

Step 2 : As M & N are homogeneous functions of same degree 3,

[Method 1 follows]

I.F. = $\frac{1}{Mx + Ny} = -\frac{1}{y^4} \neq 0$

Step 3 : Multiply the Give D.E. With the I.F.,

$$-\frac{x^2}{y^3}\,dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right) dy = 0$$

Step 4 : Clearly $M = \frac{-x^2}{y^3}$ and $N = \frac{x^3}{y^4} + \frac{1}{y}$

Step 5 : observe that $\frac{\partial M}{\partial y} = \frac{3x^2}{y^4}$ and $\frac{\partial N}{\partial x} = \frac{3x^2}{y^4}$

∴ **Exact D.E**

Step 6 : General Solution becomes → $\int M\ dx + \int \underset{\smile}{N}\ dy = C$

No "x" terms

Step 7 : $\int -\frac{x^2}{y^3}dx + \int \frac{1}{y}dy = C \quad \Rightarrow \left(-\frac{1}{y^3}\right).\frac{x^3}{3} + \log y = C$

Note : similar steps are applicable for the non-exact D.E.s which will come under Methods 2,3 and 4.

Example 2 : [Method 5] Solve $(1 + xy)\, y\, dx + (1 - xy)\, x\, dy = 0$ **:**

Note : (We can also use Method 2)

Solution : Given equation can be written as $(y\,dx + x\,dy) + (xy^2\,dx - x^2y\,dy) = 0$

Or $d\,(yx) + xy^2\,dx - x^2\,y\,dy = 0$

Dividing by x^2y^2, $\frac{d(xy)}{x^2y^2} + \frac{1}{x}dx - \frac{1}{y}dy = 0$

Integrating, $\int \frac{d(xy)}{(xy)^2} + \int \frac{1}{x}dx - \int \frac{1}{y}dy = C$

$$\Rightarrow \frac{(xy)^{-1}}{-1} + \log x - \log y = c$$

$$\Rightarrow \frac{(xy)^{-1}}{-1} + \log x - \log y = c$$

Applications of 1st order ODE

Newton's law of cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. (This law also applies to warming). If T is the temperature of the object at time t and T_s be the temperature of the surroundings, then we can formulate Newton's law of cooling as a differential equation :

$$\frac{dT}{dt} = -k(T - T_s) \quad \text{solving ,} \quad \boxed{T - T_s = c.e^{-kt}} \quad \text{where } k > 0$$

1. A cup of tea at temperature 90^0 C is placed in a room having temperature 25^0C. It cools to 60^0C in 5 minutes. Find the temperature after an interval of 5 minutes.

Solution : The problem can be classified as →

Stage 1 : T=90^0C → t=0		: C value
Stage 2 : T=60^0C → t=5 Mins		: k value
Stage 3 : T= ? → t = 10 Mins.		

This problem will come under ***Newton's law of cooling.***

Here $T_s = 25^0$C

We use the solution $\boxed{T - T_s = c.e^{-kt}}$ to solve the above problem.

Step 1 : **C** : T = 90, t = 0 $\Rightarrow$ $90 - 25 = Ce^{-k0}$ $\Rightarrow$ **C = 65.**

Step 2 : **k** : T= 60 → t = 5 Mins. $\Rightarrow$ $60 - 25 = 65.e^{-k5}$ $\Rightarrow$ $e^{-5k} = 0.53846$

$\Rightarrow$ $-5k = \ln(0.53846)$ $\Rightarrow$ **k** = 0.619/5 = **0.1238**

Step 3 : T : When t = 10 Mins, $T - 25 = 65.e^{-(0.1238)10}$ $\Rightarrow$ $T = 25 + 65.e^{-1.238}$

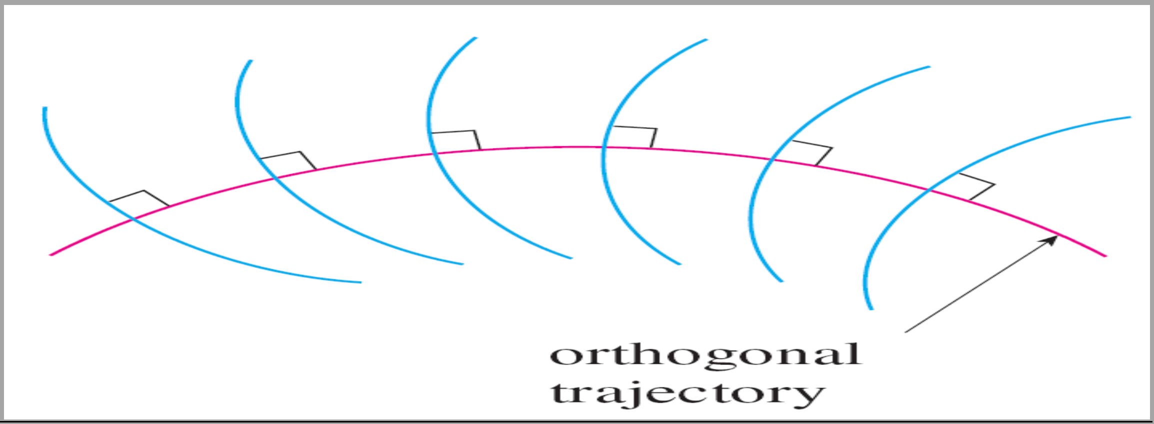
$\Rightarrow$ T = 25 + 65(0.2899)

$\Rightarrow$ $T = 44^0$ (Appx.)

Hence in 10 Mins. the temperature of the tea would be 44^0

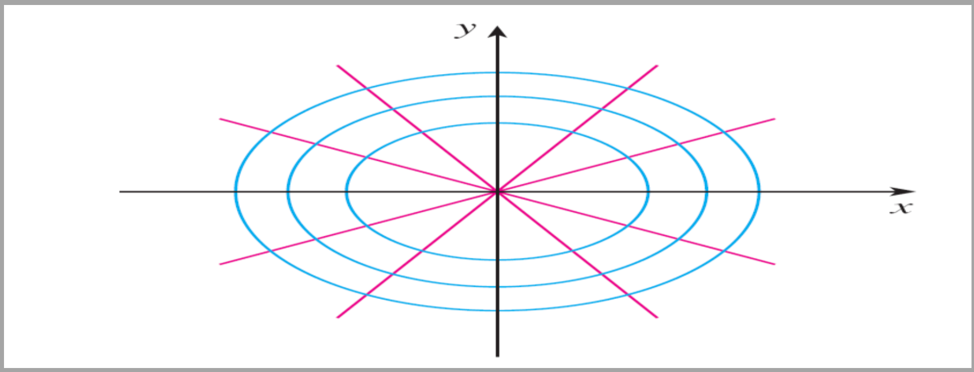
Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family orthogonally, that is, at right angles



For example, each member of the family y = mx of straight lines through the origin is an orthogonal trajectory of thefamily $x^2 + y^2 = r^2$ of concentric circles with center the origin .

We say that the two families are orthogonal trajectories of each other.



NOTE: Orthogonal trajectories has important applications in field of physics . equipotential lines and the streamlines in an irrotational 2D flow are orthogonal. In an electrostatic field,the lines of force are orthogonal to the lines of constant potential.The streamlines in aerodynamics are orthogonal trajectories of the velocity-equipotential curves.

<u>**A procedure for finding a family of orthogonal trajectories** F(x, y, C) = 0</u>

for a given family of curves F(x, y, C) = 0 is as follows:

Step 1: Determine the differential equation for the given family F(x, y, C) = 0.

Step 2: Replace y' in that equation by $-1/y'$; the resulting equation is the differential equation for the family of orthogonal trajectories.

step 3: Find the general solution of the new differential equation. This is the family of orthogonal trajectories.

<u>**Example :**</u>Find the orthogonal trajectories of the family of curves $x = ky^2$, where is k an arbitrary constant.

Solution:The curves $x = ky^2$ form a family of parabolas whose axis of symmetry is the x-axis.The first step is to find a single differential equation that is satisfied by all members of the family.If we differentiate $x = ky^2$, we get

$$1 = 2ky\frac{dy}{dx} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{1}{2ky}$$

This differential equation depends on k, but we need an equation that is valid for all values of k simultaneously.

To eliminate k we note that, from the equation of the given general parabola $x = ky^2$, we have $k = x/y^2$ and so the differential equation can be written as

$$\frac{dy}{dx} = \frac{y}{2x}$$

Or This means that the slope of the tangent line at any point (x, y) on one of the parabolas is $y' = y/(2x)$.

On an orthogonal trajectory the slope of the tangent line must be the negative reciprocal of this slope.Therefore the orthogonal trajectories must satisfy the differential equationThis differential equation is separable, and we solve it as follows:

$$\frac{y^2}{2} = -x^2 + C$$

$$x^2 + \frac{y^2}{2} = C$$

Note: In polar coordinates after getting the differential equation of the family of curves,we have to replace **$dr/d\theta$ by $-r^2\, d\theta/dr$** and then integrate the resulting differential equation

Assignment-Cum-Tutorial Questions

A. ***Objective Questions***

1. Degree and order of the D.E. $\sqrt{2\left(\frac{dy}{dx}\right)^3+4}=\left(\frac{d^2y}{dx^2}\right)^{3/2}$ are respectively ______&_____

2. Order of the differential equation $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}=c.\frac{d^2y}{dx^2}$ is ____________

3. Solution of a differential equation which is not obtained from the
4. general solution is known as ____________________
5. The necessary and sufficient condition for the D.E of the form M dx + N dy = 0 to be exact is ____________________
6. The integrating factor of M dx + N dy = 0, where M & N are homogeneous functions of same degree, is __________________
7. The integrating factor of y f(xy) dx + x g(xy) dy = 0 is __________________
8. $(xdy - ydx)/(x^2 + y^2)$ =d(________)
9. Solution of the D.E. representing Newton's law of Cooling is ___________.
10. The orthogonal trajectory is obtained by replacing dy/dx with __________
11. The orthogonal trajectory of family of curves $r=ce^{\theta}$ is __________
12. For the differential equation $(y + 3x)\,dx + xdy = 0$, the particular solution when $x = 1$, $y = 3$ is [ ]

    a) $3y^2 + 2xy = 9$ b) $3x^2 + 2y^2 = 21$ c) $3x^2 + 2y = 9$ d) $3x^2 + 2xy = 9$
13. The orthogonal trajectories of one-parameter family $x^2+2y^2=c^2$ is given by [ ]

    a) $y=ax$ b) $y^2=ax$ c) $y=ax^2$ d) $y^2=ax^2$.
14. The equation of family of curves that is orthogonal to the family of curves represented by $r\theta=c$ is givenby [ ]

    a) $r=ae^{\theta}$ b) $r=ae^{-\theta}$ c) $r=a^{\theta}$ d) $r=a^2e^{\theta^2/2}$
15. Find the integrating factor to convert non-exact D.E.(1+ xy)ydx+ (1–xy)xdy= 0 to exact D.E. [ ]

    a) $\frac{1}{2x^2y}$ b) $\frac{1}{2xy^2}$ c) $\frac{1}{2x^2y^2}$ d) $\frac{1}{2xy}$
16. Find the integrating factor to convert non-exact D.E. $2xy\,dy-(x^2+y^2+1)dx=0$ to exact D.E. [ ]

    a) y^2 b) x^2 c) $\frac{1}{y^2}$ d) $\frac{1}{x^2}$
17. Find the integrating factor to convert non-exact D.E. (y.logy) dx + (x – logy) dy = 0 to exact D.E. [ ]

    a) y b) –y c) $-\frac{1}{y}$ d) $\frac{1}{y}$
18. Which of the following equations is an exact D.E.? [ ]

    a) $(x^2 + 1)\,dx - xy\,dy = 0$ b) $x\,dy + (3x - 2y)\,dx = 0$
    c) $2xy\,dx + (2 + x^2)\,dy = 0$ d) $x^2y\,dy - y\,dx = 0$

B) Subjective Questions:

1. Solve $\left(1+e^{x/y}\right)dx+\left(1-x/y\right)e^{x/y}dy=0$
2. Solve $\left[xy\sin\left(xy\right)+\cos\left(xy\right)\right]ydx+\left[xy\sin\left(xy\right)-cos\left(xy\right)\right]xdy=0$
3. Solve: xdx + ydy = $\frac{xdy-ydx}{x^2+y^2}$
4. Solve $(x^2-ay)dx=(ax-y^2)dy$
5. Find the equation of OT of the family cruves $r^n\sin n\theta=a^n$
6. Show that the system of confocal conics $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ is self Orthogonal.
7. Find the orthogonal trajectories of the family of curves $x^2+y^2=a^2$
8. Find orthogonal trajectory of r= 2c cosθ.
9. Find orthogonal trajectory of $r^2 = a^2\cos 2\theta$
10. If the temperature of a body changes from 100°C to 70°C in 15 minutes, find when the temperature will be 40°C, if the temperature of air is 30° C.
11. The temperature of the body drops from 10^{00}C to 7^{50}C in ten minutes. When the surrounding air is at 2^{00}C temperature. What will be its temperature after half an hour? When will the temperature be 2^{50}?
12. The air is maintained at 30^0C and the temperature of the body cools from 80^0C to 60^0C in 12 minutes, find the temperature of the body after 24 minutes.

GATE QUESTIONS

1) A body originally at 60^OC cools down to 40^OC in 15 minutes when kept in air at a temperature of 25^OC. What will be the temperature of the body at the end of 30 minutes? [GATE – 2007] [ ]
   (a) 35.2^O C (b) 31.5^O C (c) 28.7^O C (d) 15^O C
2) Solution of the differential equation 3y dy/dx +2x = 0 represents a family of [GATE – 2009] [ ]
   (a)Ellipses (b) circles (c) Parabolas (d) hyperbolas
3) Match each differential equation in Group I to its family of solution curves from Group II. [GATE – 2009] [ ]

Group I	Group II
i. P. dy / dx = y/x.	1. Circles
ii. Q. dy / dx = −y/x	2. Straight lines
iii. R. dy / dx = x/y.	3. Hyperbolas
iv. S. dy / dx = −x/y	

Codes:

	P	Q	R	S
(a)	2	3	3	1
(b)	1	3	2	1
(c)	2	1	3	3
(d)	3	2	1	2

4) A D.E of the form dy/dx=f(x, y) is homogeneous if the function f(x, y) depends only on the ratio y/x or x/y [GATE:1995] [TRUE / FALSE]

5) The solution of $\frac{dy}{dx} + y^2 = 0$ is [GATE:2003] [ ]

a) $y = \frac{1}{x+c}$ b) y= $\frac{-x^3}{3}$+c c) y=ce^x d) unsolvable as equation is nonlinear

6) The solution of $\frac{dy}{dx} = y^2$ with initial value y(0)=1 bounded in the interval [GATE:2007] [ ]

a) $-\infty \leq x \leq \infty$ b) $-\infty \leq x \leq 1$ c) x<1,x>1 d) $-2 \leq x \leq 2$

7) For the D.E. $\frac{dy}{dx} + 5y = 0$ with y(0)=1 the general solution is [GATE:1994] [ ]

a) e^{5t} b) e^{-5t} c) $5\,e^{-5t}$ d) $e^{\sqrt{-5t}}$

8) Which of the following is a solution to the D.E. $\frac{dx(t)}{dt} + 3x(t) = 0$? [GATE:2008] [ ]

a) $x(t)=3e^{-1}$ b) x(t)= $2\,e^{-3t}$ c) $x(t)=\frac{-3}{2}t^2$ d) $x(t)=3t^2$

9) The order and degree of D.E. $\frac{d^3y}{dx^3} + 4\sqrt{\left(\frac{dy}{dx}\right)^3 + y^2} = 0$ are respectively [GATE:2010] [ ]

a) 3 and 2 b) 2 and 3 c) 3 and 3 d) 3 and 1

10) The solution of $\frac{dy}{dx} = x^2 y$ with the condition that y=1 at x=0 is [GATE:2007] [ ]

a) $y=e^{\frac{1}{2x}}$ b) $\ln y=\frac{x^3}{3} + 4$ c) $\ln y=\frac{x^2}{2}$ d) $y=e^{\frac{x^3}{3}}$

LINEAR ALGEBRA & DIFFERENTIAL EQUATIONS

UNIT-IV

HIGHER ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

Objectives:

- To introduce the procedure for solving second and higher order differential equations with constant coefficients and its aplications in Engineering Problems.

Syllabus:

Solving Homogeneous differential equations ,solving Non-Homogeneous differential equations when RHS terms are of the form e^{ax},sinax,cosax, polynomial in x , e^{ax} v(x) , x v(x) and Euler–Cauchy equation.

Course Out comes: At the end of the course students will be able to

- Find general solution of both homogeneous and non-homogeneous equations
- Identify and apply initial and boundary conditions to find particular solutions to second and higher order homogeneous and non_homogeneous differential equations manually and analyze and interpret the results.
- Solve applied problems encountered in engineering by formulating, analyzing differential equations of second and higher order.

Introduction:

Differential equations form the language in which the basic laws of physical science are expressed. The science tells us how a physical system changes from one instant to the next. The theory of differential equations then provides us with the tools and techniques to take this short term information and obtain the long-term overall behaviour of the system.

Definition:A D.E of the form $a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+...+a_{n-1}\frac{dy}{dx}+a_n y=Q(x)$ ______(1)

where a_0, a_1 ,..., a_n are constants and $Q(x)$ is a function of x is called a linear differential equation with constant coefficients of order n.

Definition: ***Homogeneous and non-homogeneous differential equations***

- If $Q(x)=0$ In equation(1).,it is called homogeneous differential equation with constant coefficients.

  Example: $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+3y=0$,is a second order homogeneous differential equation.

- If $Q(x)\neq 0$ in equation (1), it is called non- homogeneous differential equation With constant coefficients.

  Example: $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+3y=\sin x$ is a second order non-homogeneous differential equation

.

Note:1)$D\equiv\frac{d}{dx}$, $D^2\equiv\frac{d^2}{dx^2}$,------

Examples: $D\sin x=\frac{d}{dx}\sin x=\cos x$, $D^2\sin x=\frac{d^2}{dx^2}\sin x=-\sin x$

2) $\frac{1}{D} f(x) = \int f(x)dx$, $\frac{1}{D^2} f(x)$ = $\iint$ f(x)dx dx

Examples: $\frac{1}{D} x = \int x dx = \frac{x^2}{2}, \frac{1}{D^2} \sin x = \frac{1}{D}\left(\frac{1}{D} \sin x\right) = \frac{1}{D}\left(\int \sin x dx\right) = \frac{1}{D}(-\cos x) = -\int \cos x dx = -\sin x$

3)General solution of equation (1) = Complementary function + Particular integral

i.e., $y = y_c + y_p$

Working rule to find y_c : 1) write the given D.E in operator form as f(D)y = Q(x)

2) consider auxiliary equation f(m)=0 and find its roots

3) Depending upon the Nature of the roots we write y_c as follows:

NATURE OF ROOTS OF f(m)=0	y_c
1. m_1, m_2,……..(Real and distinct roots)	$c_1 e^{m_1 x} + c_2 e^{m_2 x}$ +----------
2. m_1, m_1, m_3----- (Two Real and equal roots)	$(c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x}$ +----------
3. a pair of imaginar$_c$ roots $m_1 = \alpha + i\beta$ $m_2 = \alpha - i\beta$	$(c_1 \cos\beta x + c_2 \sin\beta x)\, e^{\alpha x}$
4.$\alpha \pm i\beta, \alpha \pm i\beta$,$m_5$,---2 pairs of equal imaginary roots	$[(c_1 + c_2 x) \cos\beta x + (c_3 + c_4 x) \sin\beta x]\, e^{\alpha x} + c_5 e^{5x}$ +---------

Note: 1) To find y_p we have to consider

$$y_p = \frac{1}{f(D)} Q(x)$$

2)When Q(X) = 0 , $\mathbf{y_p} = 0$ i.e., in a homogeneous D.E always $\mathbf{y_p} = 0$

3)When Q(X)$\neq 0$ i.e.,in a non-homogeneous D.E following cases arises

$e^{ax}, e^{ax+b}, e^{ax-b}, a^x, k, cohax,$

$\sin ax, \cos ax, \sin(ax \pm b), \cos(ax \pm b)$ *a polynomial in x* , $e^{ax}v(x)$, $x^k v(x)$

Working rule to find y_p under case(1):

We know that

$$y_p = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$$, if f(a) $\neq 0$

Example:

$$y_p = \frac{1}{D^2+D+1} e^{-2x} = \frac{e^{-2x}}{3}$$, since f(-2) $\neq 0$

Case1) :if f(a) = 0 , then

$$y_p = \frac{1}{f(D)} e^{ax} = x\frac{1}{f'(a)} e^{ax}$$

Example: $y_p = \frac{1}{D^2+D} e^{-x} = \frac{x}{2D+1} e^{-x} = \frac{xe^{-x}}{-1}$, since f(-1) =0 and $f^1(-1) \neq 0$

Case2): if $f'(a) = 0$, then $y_p = \frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}$,if $f^{11}(a) \neq 0$ and so on

Example: $y_p = \frac{1}{(D+3)^2} e^{-3x} = \frac{x^2}{2} e^{-3x}$ ($\because$ $f^1(D)=2D+6 \Longrightarrow f^1(-3)= 0$ but $f''(D) = 2 \Rightarrow f''(-3) \neq 0$)

Working rule to find y_p under case(2) :

We know that $y_p = \frac{1}{f(D)} \sin ax$, Let us consider $f(D)=\emptyset(D^2)$ Then $y_p = \frac{1}{\phi(D^2)} \sin ax$

Casei): Now replace $D^2 = -a^2$ if $\phi(-a^2) \neq 0$

Caseii): If $\phi(D^2) = \phi(-a^2)$ =0 then we proceed as shown in below examples(3) and (4)

Example : 1) $y_p = \frac{1}{D^2-4}\cos 2x = \frac{1}{-2^2-4}\cos 2x = \frac{-1}{8}\cos 2x$

Example : 2) $y_p = \frac{1}{D^3+4}\sin 2x = \frac{1}{(-2^2)D+4}\sin 2x = \frac{(4+4D)}{(4+4D)(4-4D)}\sin 2x$

$$= \frac{(4+4D)}{16-16D^2}\sin 2x$$

$$= \frac{(1+D)}{4-4(-2^2)}\sin 2x$$

$$= \frac{(1+D)\sin 2x}{20}$$

$$= \frac{\sin 2x + 2\cos 2x}{20}$$

Example : 3) $y_p = \frac{1}{D^2+3^2}\cos 3x = \frac{x}{2D}\cos 3x = \frac{x}{2}\int \cos 3x dx = \frac{x.\sin 3x}{2.3} = \frac{x \sin 3x}{6}$

Example : 4)y_p =

$$\frac{1}{D^4-1}\sin x=\frac{x}{4D^3}\sin x=\frac{x}{4D.D^2}\sin x=\frac{x}{4D.-1^2}\sin x=\frac{x}{-4D}\sin x=\frac{-x}{4}\int\sin x dx=\frac{x\cos x}{4}$$

Note: Before finding y_p under case(3) , remember the following expansions

I. $(1+D)^{-1}$ = 1-D +D^2 –D^3 +D^4 -...................................
II. $(1-D)^{-1}$ = 1+D +D^2+D^3 +D^4 -...................................
III. $(1+D)^{-2}$ = 1-2D +3D^2 –4D^3 +5D^4 -...................................
IV. $(1-D)^{-2}$ = 1+2D +3D^2 +4 D^3 +5D^4 -...................................
V. $(1+D)^{-3}$ = 1-3D +6D^2 –10D^3 +...................................
VI. $(1-D)^{-3}$ = 1+3D +6D^2 +10D^3 +...................................

Working rule to find y_p under case(3):

We know that $y_p = \frac{1}{f(D)}Q(x)$,where $Q(x)$ is a polynomial in x

convert $\frac{1}{f(D)}$ into $(1+\psi)^{-1}$ where ψ is a function of D's ,then using above expansions we get y_p

Example : 1) Consider $y_p = \frac{1}{D^2+3}x^2$

$$=\frac{1}{3}\frac{1}{(1+\frac{D^2}{3})}x^2$$

$$=\frac{1}{3}\left(1+\frac{D^2}{3}\right)^{-1}x^2$$

$$=\frac{1}{3}\left(1-\frac{D^2}{3}+\left(\frac{D^2}{3}\right)^2-....\right)x^2$$

$$=\frac{1}{3}\left(x^2-\frac{2}{3}\right)$$

Example : 2) $y_p = \frac{1}{D^3-4D}3x^2$

$$=\frac{1}{D(D^2-4)}3x^2$$

$$=\frac{3}{D}\frac{1}{-4(1-\frac{D^2}{4})}x^2$$

$$=\frac{-3}{4D}\left(1-\frac{D^2}{4}\right)^{-1}x^2$$

$$=\frac{-3}{4D}\left(1+\frac{D^2}{4}+\frac{D^4}{16}+---\right)x^2$$

$$=\frac{-3}{4D}\left(x^2+\frac{1}{2}\right)$$

$$= \frac{-3}{4}\left(\frac{x^2}{D} + \frac{1}{2D}\right)$$

$$= \frac{-3}{4}\left(\frac{x^3}{3} + \frac{x}{2}\right)$$

Working rule to find y_p under case(4):

We know that $y_p = \frac{1}{f(D)} Q(x)$

$$= \frac{1}{f(D)} e^{ax} v(x)$$

$$= e^{ax} \frac{1}{f(D+a)} v(x)$$

Depending on the nature of V(x) solve it further

Example: 1)y_p = $\frac{1}{D+2} e^{3x} x$

$$= e^{3x} \frac{1}{(D+3)+2} x$$

$$= e^{3x} \frac{1}{D+5} x$$

$$= e^{3x} \frac{1}{5(1+\frac{D}{5})} x$$

$$= e^{3x} \frac{1}{5}\left(1+\frac{D}{5}\right)^{-1} x$$

$$= e^{3x} \frac{1}{5}\left(1-\frac{D}{5}+\frac{D^2}{5^2}-.....\right)x$$

$$= \frac{e^{3x}}{5}\left(x-\frac{1}{5}\right)$$

Example : 2) $y_p = \frac{1}{D^2-6D+13} 8e^{3x} \sin 2x$

$$= 8e^{3x} \frac{1}{(D+3)^2-6(D+3)+13} \sin 2x$$

$$= 8e^{3x} \frac{1}{D^2+4} \sin 2x$$

$$=8e^{3x} . \frac{x}{2D} \sin 2x$$

$$=8e^{3x} .- \frac{x}{4} \cos 2x$$

$$= -2 xe^{3x} \cos 2x$$

Working rule to find y_p under case(5):

We know that $y_p=\frac{1}{f(D)}Q(x)=\frac{1}{f(D)}x^k v(x)$ **Note:** $e^{i\theta}=\cos\theta + i\sin\theta$

R.P of $e^{i\theta}$ I.P of $e^{i\theta}$

Case(1): Let k = 1 then $y_p=\left[x-\frac{f^1(D)}{f(D)}\right]\frac{1}{f(D)}v(x)$

Case(2): i) Let $k \neq 1$ and $v(x)=\sin ax$

$$Y_p=\frac{1}{f(D)}x^k \sin ax$$
$$=\frac{1}{f(D)}x^k \text{ I.P of } e^{iax}$$
$$=\text{I.P of } \frac{1}{f(D)}x^k e^{iax}$$
$$=\text{I.P of } e^{iax}\frac{1}{f(D+ia)}x^k$$

By using previous related method we will solve it

finally replace e^{iax} =cosax+ i sinax

ii) Let $k \neq 1$ and $v(x)=\cos ax$

$$Y_p=\frac{1}{f(D)}x^k \cos ax$$
$$=\frac{1}{f(D)}x^k \text{ R.P of } e^{iax}$$
$$=\text{R.P of } \frac{1}{f(D)}x^k e^{iax}$$
$$=\text{R.P of } e^{iax}\frac{1}{f(D+ia)}x^k$$

By using previous related method we will solve it

finally replace e^{iax} =cosax+ i sinax

Example: $Y_p=\frac{1}{D^2}x\sin 2x$

$$=\frac{1}{D^2}x\ I.P of\ e^{i2x}$$
$$=\text{I.P of } \frac{1}{D^2}xe^{i2x}$$
$$=\text{I.P of } e^{i2x}\frac{1}{(D+2i)^2}x$$
$$=\text{I.P of } e^{i2x}\frac{1}{-4(1+\frac{D}{2i})^2}x$$
$$=\text{I.P of } \frac{-e^{i2x}}{4}.\left(1+\frac{D}{2i}\right)^{-2}x$$
$$=\text{I.P of } \frac{-e^{i2x}}{4}.\left(1-2\frac{D}{2i}+3\frac{D^2}{(2i)^2}----\right)x$$
$$=\text{I.P of } \frac{-e^{i2x}}{4}.\left(x-\frac{1}{i}\right)$$

$= \quad$ I.P of $\frac{-e^{i2x}}{4}.(x+i)$

$= \quad$ I.P of $\left(\frac{-\cos 2x - i\sin 2x}{4}\right)(x+i)$

$= \quad \frac{-\cos 2x}{4} - \frac{x\sin 2x}{4}$

$= -\frac{1}{4}(\cos 2x + x\sin 2x)$

Euler-Cauchy's linear equation:

An equation of the form $x^n \frac{d^n y}{dx^n} + p_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + p_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}}$ +---------+$p_n y = \emptyset(x)$-----(1)
Where p_1, p_2,------------------------------- p_n are real constants and $\emptyset(x)$ is a function of x is called homogeneous linear equation or Euler-Cauchy's linear equation of order n.
Operator of is Euler-Cauchy's linear equation of order n is
$(x^n D^n + p_1 x^{n-1} D^{n-1} + p_2 x^{n-2} D^{n-2}$+-----+$p_n)y = \emptyset(x)$.

Working rule to solve Euler-Cauchy's linear equation:

Let x = e^z or logx = z, x>0

$\frac{dz}{dx} = \frac{1}{x}$, $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$

$x\frac{dy}{dx} = \frac{dy}{dz}$

Similarly we can get

$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$

$x^3 \frac{3y}{dx^3} = \frac{3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}$

Let $\theta = \frac{d}{dz}$ xD = θ, x$D^2 = \theta(\theta-1)$, x$D^3 = \theta(\theta-1)(\theta-2)$,--------.

By substituting these in equation (1) ,it becomes a linear differential equation with constant coefficients.This can be solved as earlier methods

Assignment-Cum-Tutorial Questions

A. Questions testing the remembering / understanding level of students

I) Objective Questions:

1. Solution of $(D^2 - a^2)y = 0$ is ______________________
2. The general solution of the D.E. $(D^4 - 6D^3 + 12D^2 - 8D)y = 0$ is ______________
3. Solution of $D^3 y = 0$ is ___________________________
4. The particular integral of $(D^2 + 4^2)y = \sin 6x$ is ________________________
5. $\frac{1}{D^2}x^2 =$__________________
6. $D^2(2x + 4) =$ _________________
7. The complete solution of the equation f(D) y = Q(x) is _____________
8. Roots of Auxiallary equation $m^4 + 4 = 0$ are _____________
9. $\frac{1}{f(D^2)}\sin ax =$________________
10. The real and imaginary part of x^2e^{i3x} is _____ and ______ respectively
11. $\frac{1}{f(D)}e^{ax}v(x) =$________________
12. Roots of auxiliary equation $m^2(m^2 + 4) = 0$ are _________
13. Y_p of $\frac{1}{D^2 + 2D}e^{-2x} =$ ________________________________
14. In a homogenous linear D.E. f(D) y = 0 , the general solution of y is ______________
15. In a non-homogenous linear D.E. f(D) y = Q(x) , then the general solution of y is _______
16. $\frac{1}{D - a}e^{ax} =$ _______________
17. $\frac{1}{D^2 - 5D}x =$ _________________
18. P.I. of $\frac{1}{f(D)}xv(x) =$ ________________________
19. P.I of $(D-1)^2 y = e^x \sin x$ is ________
20. The solution of the D.E $(D^2 - 2D + 5)^2 y = 0$ is__________________
21. The solution of the differential equation $y'' + y = 0$ satisfying the conditions y(0)=1 and $y(\pi/2)=2$ is ___________
22. The general solution of $(4D^3 + 4D^2 + D)y = 0$ is ___________________
23. P.I. of $\frac{e^{-x}}{D^2 + D + 1}$ is _________________

II) Descriptive Questions:

1. Obtain the general solution of $(D-2)(D+1)^2y = 0$.
2. Find particular solution of initial value problem $y''+2y'+2y=0$ with $y(0) = 1$ $y^1(0) = -1$
3. List out the general properties of solutions of linear ODE's.
4. It is given that $y''+2y' = y = 0,$ with $y(0) = 0$, $y(1)=0$ then what is $y(0.5)$?
5. Given that $x''+3x=0$ and $x(0) = 1$, $x'(0) = 0$ then what is $x(1)$.
6. Solve: $(D^2 - 4D + 3)y = \sin 3x \cos 2x$
7. Solve $(D^2 - 1)y = 2e^x + 3x$
8. Solve $(D^2-2D+1)\, y = x\, e^x \sin x$.
9. Solve $(4D^2-4D+1)\, y = 100$.
10. Give examples of C.F. for different nature of roots of an auxiliary equation.
11. Solve $(D^3-5D^2+8D-4)\, y = e^{2x.}$
12. Solve $(D^4-4D+4)\, y = e^{2x} + x^2 + \sin 3x$.
13. Solve $(D^2 - 4D + 4)\, y = 8x^2 e^{2x} \sin 2x$.
14. Solve $(D^3+1)\, y = \cos(2x-1)$
15. Solve $y''-y'-2y=3e^{3x}$, $y(0) = 0$, $y^1(0) = 2$
16. Solve $(D+2)(D-1)^2\, y = 2 \sinh x$
17. Find y of $(D^3 - 7D^2 + 14D - 8)y = e^x \cos 2x$
18. Solve $(x^2D^2-xD+1)y = \log x$.
19. Solve $(x^2D^2-3xD+4)y = (1+x)^2$.
20. Solve $(x^2D^2-xD+2)y = x \log x$.

B. Question testing the ability of students in applying the concepts.

I) Multiple Choice Questions:

1. Solution of $(D^3 + D)\, y = 0$ is [ ]

   a) $y = A\cos x + B\sin x$ b) $y = Ae^x + Be^{-x}$ c) $y = A + Be^x + Ce^{-x}$ d) $y = A + B\cos x + C\sin x$

2. Solution $(D^3 - D^2)y = 0$ is [ ]

   a) $y = Ae^x + B$ b) $y = (A + Bx)\, e^x + C$ c) $y = A + Bx + Ce^x$ d) none

3. P.I. of $\left(\frac{1}{D^2+1}\right)\cos^2 x =$ [ ]

   a) $\cos x$ b) $-\cos x$ c) $\sin x$ d) $-\sin x$

4. General solution of $(D^2 - 1)y = x^2 + x$ is [ ]

   a) $y = Ae^x + Be^{-x} + (x^2 + x + 2)$ b) $y = Ae^x + Be^{-x} - (x^2 + x + 2)$

   c) $y = Ae^x + Be^{-x} + 1$ d) $y = A\cos x + B\sin x - 1$

5. P.I. of $(D + 1)^2\, y = e^{-x} \cdot x$ is [ ]

   a) $e^{-x} \cdot \frac{x^2}{2}$ b) $e^{-x} \cdot \frac{x^3}{6}$ c) $e^{-x} \cdot \frac{x^4}{24}$ d) $\frac{e^{-x}}{24}$

7. A two variable (one is dependent and other is independent) D.E., with initial condition , geometrically represents [ ]

   a) Particular curve b) set of curves from family of curves
   c) Particular surface d) set of surfaces from family of surfaces

8. Every D.E.(without initial or boundary condition) must have ___ [ ]

   a) Particular solution b) Singular solution c) General solution d) None of these

9. The auxiliary equation of a higher order L.D.E. is _____________ [ ]

   a) A polynomial equation of degree higher than the order of the D.E.
   b) A polynomial equation of degree equals to the order of the D.E.
   c) A polynomial equation of degree lower than the order of the D.E.
   d) None

10.The complimentary function of the general solution of a linear D.E. depends on [ ]

a) Only the nature of the roots of the auxiliary equation
b) Both the nature and repetition of roots
c) Only the number of unequal roots
d) None

11. The solutions of a D.E. are the general solution, a particular solution and the singular solution where [ ]

a) The singular solution is obtained from the general solution by choosing suitable values of constants
b) A particular solution may be similar to the singular solution
c) The singular solution does not satisfy the given D.E.
d) The singular solution must satisfy the given D.E.

12.Particular integral of (D^2+9) y = cosx is_________ [ ]

a) $\frac{\cos x}{8}$ b) $\frac{\sin x}{8}$ c) $\frac{\cos x}{10}$ d) $\frac{\sin x}{10}$

13.The complementary function of (D^3+D) y = 5 is ________ [ ]

a)a+bcosx+csinx b)bcosx+csinx c)a+bcosx d)none

14. C.F of $(D^2+4D+13)y = e^{-2x}\sin 3x$ is __________ [ ]

a) $A\sin 3x + B\cos 3x$ b) $e^{-3x}(A\cos 2x + B\sin 2x)$ c) $e^{-2x}(A\cos 3x + B\sin 3x)$ d)none

15. $\frac{1}{(D-2)^3}e^{2x} =$ ________________ [ ]

a) $\frac{x^2e^{2x}}{6}$ b) $\frac{x^3e^{2x}}{6}$ c) $\frac{x^2e^{2x}}{4}$ d)none

16.The particular integral of $(D^2-4)y = \sin 3x$ is ____________ [ ]

a) $\frac{1}{4}$ b) $\frac{-1}{13}$ c) $\frac{1}{5}$ d)None

17. $e^{-x}\left(a\cos\sqrt{3x} + b\sin\sqrt{3x}\right) + ce^{2x}$ is the general solution of [ ]

a) $(D^3+4)y = 0$ b) $(D^3-8)y = 0$ c) $(D^3+8)y = 0$ d) $(D^3-2D^2+D-2)y = 0$

Problems :

1. Apply related case to find solution of $(D^4+2D^2+1)y = x\cos^2 x$
2. Apply related case to find solution of $(D^2+4)y = x\sin x$
3. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$
4. Solve $(D^4+10D^2+9)y = 96\sin 2x\cos x$
5. Solve $(D^2-2D+4)y = e^x \sin\frac{x}{2}$
6. Show that the complete solution of $(D^4+2D^2+1)y = x^2\cos x$ is
7. $y = (a+bx)\cos x + (c+dx)\sin x + \frac{1}{48}\left(4x^3\sin x - x^2(x^2-9)\cos x\right)$
8. Show that the particular integral of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$ is

$$\frac{e^{3x}}{2}\left(x - \frac{3}{2}\right) + \frac{1}{20}(3\cos 2x - \sin 2x).$$

Linear Algebra & Integral Transforms

Unit – V (Partial differentiation)

Course Objectives:

- To introduce the concept of total derivative, Jacobian & maxima and minima

Syllabus:

Total Derivative – chain Rule – Functional Dependence – Jacobian – Application – Maxima and Minima of functions of two / three variables

Course Out comes:

At the end of the course students will be able to

- Find total derivative of the given function
- Verify the functional dependence of functions
- Find maxima and minima of functions of two / three variables

Learning Material

Partial Differentiation:-

Let z = f (x,y) be a function of two variables x and y.

Then $\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$, if it exists is said to be partial derivative of z of f (x, y) w.r.t "x";

It is denoted by the symbol $\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x} or f_x$ i.e. The partial derivative of z = f (x,y) with respect to "x"is done, y is kept constant.

Similarly the partial derivative of z = f(x,y) wrt "y" keeping "x" constant is defined by

$\lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$ and it is denoted by $\frac{\partial z}{\partial y} or\, f_y$

In the same way, the partial derivatives of the function z = f(x_1, x_2………x_n) w.r.t "x_i" keeping other variables constant can be defined by

$$\frac{\partial z}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2....x_i + \Delta x_i....x_n) - f(x_1, x_2....x_i......x_i)}{\Delta x_i}, i = 1,2....n$$

Higher Order Partial Derivatives:-

In general the first order partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are also functions of x and y and they can be differentiated repeatedly to get higher order partial derivatives.

So $\frac{\partial}{\partial x} = \left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$, $\quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$

$\frac{\partial}{\partial x} = \left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}$, $\quad \left(\frac{\partial}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

<u>Model Problems</u>:

1. The plane x = 1 intersects the parabdoid z = x^2 + y^2 in a parabola; Find the slope of the Tangent to the parabola at (1,2,5)?

   Ans: The slope is the value of the partial derivative $\frac{\partial z}{\partial y} at (1,2)$

   (1) $\therefore z = x^2 + y^2$; $\quad$ (2) $\frac{\partial z}{\partial y} = 2y$ $\quad$ (3) $\left(\frac{\partial z}{\partial y}\right) = 2\times 2 = 4$

2. If u = $(1-2xy + y^2)^{-1/2}$ then show that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = u^3 y^2$

   Sol: $\frac{\partial u}{\partial x} = -\frac{1}{2}\left[1-2xy+y^2\right]^{-3/2} \times (-2y)$

$$\frac{\partial u}{\partial y}=-\frac{1}{2}\left[1-2xy+y^2\right]^{-3/2}\times[-2x+2y]=(x-y)(1-2xy+y^2)^{-3/2}$$

$$\Rightarrow\therefore x\times\left[+y\left(1-2xy+y^2\right)^{-3/2}\right]-y\left[\left(1-2xy+y^2\right)^{-3/2}\times(x-y)\right]$$

$$\Rightarrow\left(1-2xy+y^2\right)^{-3/2}=\left(1-xy+y^2\right)^{-3/2}y^2$$

$= u^3 y^2$

3. u (x,t) = a^{-gx} $e^{\sin(nt-gx)}$ where a,g,n are constants, satisfying the equation $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$ prove that

$$g=\frac{1}{a}\sqrt{\frac{n}{2}}$$

Sol: u (x, t) = $a^{-gx}e^{\sin(nt-gx)}$

$$LHS=\frac{\partial u}{\partial t}=a^{-gx}e^{\cos(nt-gx)}\times n$$

$$=ane^{-gx}\cos^{(nt-gx)}$$

Diff. w.r.t "x" we get

$$\frac{\partial u}{\partial x}=ae^{-gx}\exp(\cos(nt-gx))\times(-g)+\sin(nt-gx)a(-g)e^{-gx}$$

$$=(-ag)\left[e^{-gx}\cos(nt-gx)+\sin(nt-gx)e^{-gx}\right]$$

$$\frac{\partial^2 u}{\partial x^2}=2e^{-gx}\times\cos(nt-gx)\times g^2\times a$$

But $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$

$$\cos(nt-gx)\times an\times e^{-gx}=2e^{-gx}\cos(nt-gx)\times g^2\times a$$

$$an=2\cos(nt-gx)\times a\times g^2$$

$$n=2g^2\Rightarrow g^2=\frac{n}{2}\Rightarrow g=\sqrt{\frac{3}{2}}$$

Total Derivative:

If u = f (x,y), where x = $\varphi(t)$, $y=\psi(t)$ then we express u as a function of t alone by substituting the values of x and y in f(x,y); thus we can find ordinary derivative $\frac{du}{dt}$ is called the <u>total derivative of u</u> to distinguish it from the partial derivatives $\frac{\partial u}{\partial x}$ *and* $\frac{\partial u}{\partial y}$.

Chain Rule:

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\times\frac{dy}{dt}$$

$$=\frac{\partial u}{\partial x}\times\frac{dx}{dt}+\frac{\partial u}{\partial y}\times\frac{dy}{dt}-----------------(1)$$

In three variables we get when u = f(x,y,z)

Where x, y,z are all functions of a variable t, then $\frac{du}{dt}=\frac{\partial u}{\partial x}\times\frac{dx}{dt}+\frac{\partial u}{\partial y}\times\frac{dy}{dt}+\frac{\partial u}{\partial z}\times\frac{dz}{dt}$

Differentiation of implicit functions:-

If f (x,y) = c be an implicit relation between x and y which defines as a differentiable function of x when t = x in (1), it becomes

In implicit function (2) becomes

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\times\frac{dy}{dx}------------------(2)$$

$$0=\frac{df}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\times\frac{dy}{dx}$$

$$\therefore\frac{df}{dx}-\frac{\partial f}{\partial x}-\frac{\partial f/\partial x}{\partial f/\partial y}=\frac{dy}{dx}$$

4. Show that $\frac{\partial x}{\partial u}=\frac{1}{r}\frac{\partial y}{\partial\theta}$; $\frac{\partial y}{\partial u}=-\frac{1}{r}\frac{\partial x}{\partial\theta}$ and hence show that $\frac{\partial^2 x}{\partial r^2}+\frac{1}{r}\frac{\partial x}{\partial r}+\frac{1}{r^2}\frac{\partial^2 x}{\partial\theta^2}=0$

If $x=e^{r\cos\theta}.cis(r\sin\theta)$

$y=e^{r\cos\theta}.\sin(r\sin\theta)$

Sol: x = $e^{r\cos\theta}\cos(r\sin\theta)$

$$\frac{\partial x}{\partial r}=e^{r\cos\theta}[-\sin(r\sin\theta)\times\sin\theta]+[\cos(r\sin\theta)]\times e^{r\cos\theta}\times\cos\theta$$

$$=e^{r\cos\theta}[-\sin\theta\,\sin(r\sin\theta)]+\cos\theta\,\cos(r\sin\theta).$$

$$=e^{r\cos\theta}[\cos\{\theta+r\sin\theta\}]------------------(1)$$

$$y=e^{r\cos\theta}\sin(r\sin\theta)$$

$$\frac{\partial y}{\partial r}=e^{r\cos\theta}\times\cos(r\sin\theta)\times\sin\theta+\sin(r\sin\theta)e^{e\cos\theta}\cos\theta$$

$$=e^{r\sin\theta}[\sin\theta\times\cos(r\sin\theta)+\cos\theta\times\sin(r\sin\theta)]$$

$$=e^{r\sin\theta}[\sin(\theta+r\sin\theta)]------------------(2)$$

$$\frac{\partial x}{\partial\theta}=e^{r\cos\theta}[-\sin(r\sin\theta)\times r\cos\theta]+\cos(r\sin\theta)\times e^{r\cos\theta}\times(-r\sin\theta)$$

$$=-re^{r\cos\theta}[+\cos\theta\sin(r\sin\theta)+\sin\theta\cos(r\sin\theta)]$$

$$=re^{r\cos\theta}[\sin(\theta+r\sin\theta)]----------------(3)$$

$$\frac{\partial y}{\partial\theta}=e^{r\cos\theta}[\cos(r\sin\theta)\times r\cos\theta+\sin(r\sin\theta)e^{r\cos\theta}\times(-e\sin\theta)]$$

$$=re^{r\cos\theta}[\cos\theta\,\cos(r\sin\theta)-\sin\theta\sin(r\sin\theta)]$$

$$=re^{r\cos\theta}\cos(\theta+r\sin\theta)----------------(4)$$

To show that $\frac{\partial x}{\partial r}=\frac{1}{r}\frac{\partial y}{\partial\theta}$

$$e^{r\cos\theta}\cos(\theta+r\sin\theta)$$

$$=\frac{1}{r}\times[re^{r\cos\theta}\cos(\theta+r\sin\theta)]$$ equal

To show that $\frac{\partial y}{\partial r}=-\frac{1}{r}\frac{\partial x}{\partial\theta}$

$$e^{r\cos\theta}\sin[\theta+r\sin\theta]$$

$$=-\frac{1}{r}[-r^{r\cos\theta}e\sin(\theta+r\sin\theta)]$$

$$=e^{r\cos\theta}\sin(\theta+r\sin\theta)$$

Simple Method:-

$$\frac{\partial^2 x}{\partial x^2}=\frac{\partial}{\partial u}\left(\frac{1}{r}\frac{\partial y}{\partial \theta}\right)=\frac{1}{r}\frac{\partial^2 y}{\partial r\partial \theta}+\left(\frac{\partial y}{\partial \theta}\right)\left(-\frac{1}{r^2}\right)$$

$$\frac{1}{r^2}\frac{\partial^2 x}{\partial \theta^2}=\frac{1}{r^2}\left[\frac{\partial}{\partial \theta}\left(-r\frac{\partial y}{\partial r}\right)\right]=-r\frac{\partial^2 y}{\partial \theta 0 r}$$

$$\frac{1}{r}\frac{\partial^2 y}{\partial r\partial \theta}-\frac{1}{r^2}\frac{\partial y}{\partial \theta}+\frac{1}{r}\frac{\partial x}{\partial r}-r\frac{\partial^2 y}{\partial x\partial \theta}\times\frac{1}{r^2}$$

$$\frac{1}{r}\frac{\partial^2 y}{\partial r\partial \theta}-\frac{1}{r^2}\frac{\partial y}{\partial \theta}+\frac{1}{r}\frac{\partial x}{\partial r}-\frac{1}{r}\frac{\partial^2 y}{\partial r\partial \theta}$$

$$-\frac{1}{r^2}\frac{\partial y}{\partial \theta}+\frac{1}{r}\times\frac{1}{r}\frac{\partial y}{\partial \theta}=0$$

Jacobians:-

If u and v are functions of two independent variables x and y, then the determinant $\begin{bmatrix}\partial u/\partial x & \partial u/\partial y\\ \partial v/\partial x & \partial x/\partial y\end{bmatrix}$ is called the Jacobian of u, v with respect to x,y and is written as $\frac{\partial(u,v)}{\partial(x,y)} or J\left(\frac{u,v}{x,y}\right)$

Similarly the Jacobian of u, v, w with respect to x,y, z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\partial u/\partial x & \partial u/\partial y & \partial u/\partial z\\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z\\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z\end{vmatrix}=\begin{bmatrix}u_x & u_y & u_z\\ v_x & v_y & u_z\\ w_x & w_y & w_z\end{bmatrix}$$

Properties of Jacobians:-

(1) If $J=\frac{\partial(u,v)}{\partial(x,y)} and\ J^1=\frac{\partial(x,y)}{\partial(u,v)}$ $then$ $S.T\ JJ^1=1$

Proof: Let u = f (x,y) and v = g (x,y)

After solving for x and y, suppose we have $x=\varphi(u,v)\ and\ y=\psi(u,v)$ thus

$$\frac{\partial u}{\partial u}=1=\frac{\partial u}{\partial x}\times\frac{\partial x}{\partial u}+\frac{\partial u}{\partial y}\times\frac{\partial y}{\partial u}$$

$$\frac{\partial u}{\partial v}=0=\frac{\partial u}{\partial x}\times\frac{\partial x}{\partial v}+\frac{\partial u}{\partial y}\times\frac{\partial y}{\partial v}$$

$$\frac{\partial v}{\partial u}=0=\frac{\partial v}{\partial x}\times\frac{\partial x}{\partial u}+\frac{\partial v}{\partial y}\times\frac{\partial y}{\partial u}$$

$$\frac{\partial v}{\partial v}=1=\frac{\partial u}{\partial x}\times\frac{\partial x}{\partial v}+\frac{\partial u}{\partial y}\times\frac{\partial y}{\partial v}$$

$$\therefore\frac{\partial(u,v)}{\partial(x,y)}\times\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial x}{\partial y}\end{vmatrix}\times\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$$

$$=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial x}{\partial y}\end{bmatrix}\times\begin{bmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial v}{\partial v} & \frac{\partial y}{\partial v}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=1$$

Property:-

If u, v are functions of r,s and r,s are functions of x,y then S.T. $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$

Sol: Consider RHS

$$\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \begin{bmatrix} \partial u/\partial r & \partial u/\partial s \\ \partial v/\partial r & \partial v/\partial s \end{bmatrix} \times \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial s}+\frac{\partial u}{\partial s}\times\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\times\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\times\frac{\partial u}{\partial x}+\frac{\partial v}{\partial s}\times\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial y}+\frac{\partial v}{\partial s}\times\frac{\partial s}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \frac{\partial(u,v)}{\partial(x,y)} = LHS$$

Model Problem:

Sol: $u = \frac{yz}{x}$ $\quad \frac{\partial u}{\partial x} = u_x = -\frac{yz}{x^2}$ $\quad \frac{\partial u}{\partial y} = \frac{u}{y} = z/x$ $\quad \frac{\partial u}{\partial z} = u_z = y/x$

$v = \frac{xz}{y}$ $\quad \frac{\partial v}{\partial x} = z/y$ $\quad \frac{\partial v}{\partial y} = -\frac{xz}{y^2}$ $\quad \frac{\partial v}{\partial z} = x/y$

$w = \frac{xy}{z}$ $\quad \frac{\partial w}{\partial x} = \frac{y}{z}$ $\quad \frac{\partial w}{\partial y} = \frac{x}{z}$ $\quad \frac{\partial w}{\partial z} = \frac{-xy}{z^2}$

$$\partial\left(\frac{u,v,w}{x,y,z}\right) = \begin{bmatrix} -yz/x^2 & z/x & y/x \\ z/x & -xz/y^2 & x/y \\ y/x & x/y & -xy/z^2 \end{bmatrix}$$

Multiply C_1 with x

C_2 with y

C_3 with z

$$= \begin{bmatrix} -yz/x^2 & z/x & y/x \\ z/x & -xz/y^2 & x/y \\ y/x & x/y & -xy/z^2 \end{bmatrix} \Rightarrow \frac{1}{y_z}\begin{bmatrix} -xyz/x^2 & yz/x & yz/x \\ xz/y & -xyz/y^2 & xz/y \\ xy/x & yx/y & -xyz/z^2 \end{bmatrix}$$

$$= \frac{1}{xyz}\times\frac{yz}{x}\times\frac{xz}{y}\times\frac{xy}{z}\times\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Taking common $\frac{yz}{x}$ from R_1

$\frac{xz}{y}$ from R_2

$\frac{xy}{z}$ from R_3

$$=\frac{x^2y^2z^2}{(xyz)^2}[-1(1-1)-1(-1-1)+1(1+1)]$$

= -1 (-2) +1 x 2 = 4

Functional Dependence: -

If u = f (x,y) and v = g (x,y) are two given differentiable functions in the dependent variables x,y; suppose these functions are connected by a relation F (u, v) = 0 where F is differentiable.

We say that u and v functionally dependent on one another , if the partial derivatives u_x, u_y, v_x, v_y are all not zero at a time.

Theorem:-

If the functions u and v of the independent variable x and y are functionally dependent then the Jacobian vanishes.

Proof:-Consider F (u, v) = 0

Differentiating F (u, v) = 0 partially wrt "x and y, we get $\frac{\partial F}{\partial u}\times\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\times\frac{\partial v}{\partial x}=0$

$$\frac{\partial F}{\partial u}\times\frac{\partial u}{\partial y}+\frac{\partial F}{\partial v}\times\frac{\partial v}{\partial y}=0$$

A Non-trivial solution $F_u \neq 0$; $F_v \neq 0$, to this system exists if the coefficient determinant is zero.

$$\Rightarrow \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 0 \ i.e. \ \frac{\partial(u,v)}{\partial(x,y)}=0$$

Note:- If the Jacobian $J\left(\frac{u,v}{x,y}\right)=0$ then u and v are said to be functionally independent.

Model Problems:

Show that the functions u = xy + yz + zx,v = $x^2 + y^2 + z^2$ and w = x + y + z are functionally related. Find the relation between them?

Sol:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}=\begin{bmatrix} y+z & x+z & y+x \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{bmatrix}$$

u = xy + yz + zx	$u_x = y + z$	$u_y = x + z$	$u_z = y + x$
v = $x^2 + y^2 + z^2$	v = 2x	$v_y = 2y$	$w_x = 1$
w = x - y + z	$w_x = 1$	$w_y = 1$	$w_z = 1$

$R_1 + R_2$

$$=2\times\begin{bmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ 1 & 1 & 1 \end{bmatrix}=2(x+y+z)\begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{pmatrix}$$

$$=2\begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ 0 & 0 & 0 \end{bmatrix}_{R_3-R_1} \quad =0$$

u,v,w are functionally dependent $\Rightarrow$ Functional relationship exists among u,v,w.

Now $w^2 = (x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$

= v + 2u

$$\therefore w^2 = v + 2u$$

Maxima and Minima values of f (x,y)

Working Rule to find the Maximum and Minimum values of f (x,y):-

- Find $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ and equate each to zero. Solve these as simultaneous equations in x and y. Let (a,b) (c,d) be the pairs of values.
- Calculate the value of $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ for each pair of values.
- (i) If $rt - s^2 > 0$ and $r < 0$ at (a,b), f (a,b) is a Max. value
  (ii) If $rt - s^2 > 0$ at (a,b), f (a,b) is a Mini value
  (iii) If $rt - s^2 < 0$ at (a,b), f (a,b) is not an extreme value. i.e. (a,b) is a saddle point.
  (iv) If $rt - s^2 = 0$ at (a,b), the case is doubtful and needs further investigation.

Model Problems:-

Examine the following function for extreme values?

Sol: $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$f_x = 4x^3 - 4x + 4y$ $\qquad$ $f_y = 4y^3 + 4x - 4y$

$f_{xx} = 12x^2 - 4 = r$ $\qquad$ $f_{yy} = t = 12y^2 - 4$

$f_{xy} = s = 4$

Now If $f_x = 0$ $\qquad$ if $f_y = 0$

$x^3 - x + y = 0$ $\qquad$ $y^3 - y + x = 0$

$y^3 + x - y = 0$

$x^3 + y^3 = 0 \Rightarrow (x+y)\left[x^2 - xy + y^2\right] = 0$

$\Rightarrow x = -y$

Putting $x = -y$ in $f_x = 0 \Rightarrow x^3 - x - x = 0$

$x^3 - 2x = 0$

$x^2 - 2 = 0 \Rightarrow x^2 = 2$

$x = \pm\sqrt{2}$

$y = \mp\sqrt{2}$

(i) At $(\sqrt{2}, -\sqrt{2})$, $rt - s^2 = \left[12(\sqrt{2})^2 - 4\right][12\times 2 - 4] - 4^2$

$= 20 \times 20 - 4^2 = 400 - 16 = 384 > 0.$

Hence $f(\sqrt{2}, -\sqrt{2})$ is a min value.

At $(\sqrt{2}, -\sqrt{2}) \Rightarrow rt - s^2 = \left[12(-\sqrt{2})^2 - 4\right]\left[12(\sqrt{2})^2 - 4^2\right]0$ *and* $r = 12(-\sqrt{2})^2 - 4 > 0$

Hence f $(-\sqrt{2}, \sqrt{2})$ is also a min value.

(ii) At (0,0) $rt - s^2 = [12 \times 0^2 - 4]\,[12 \times 0^2 - 4] - 4^2$

$= (-4)(-4) - 4^2 = 0$

$\therefore$ Further investigation is needed.

(iii) Now f (0,0) = 0 and for points along the x – Axis where y = 0, $f(x,y) = x^4 - 2x^2 = x^2(x^2 - 2)$ which is negative for points in the neighborhood of the origin.

Thus in the neighborhood of (0,0) there are points

When $f(x,y) < f(0,0)$ and there are points where $f(x,y) > f(0,0)$

Hence f (0,0) is not <u>Extreme Value</u> i.e. it is a saddle point.

Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube?

Sol:

Let 2x, 2y, 2z be the length, breadth and height of the rectangular solid so that its volume V = 8xyz

Let R be the redius of the sphere so that $x^2 + y^2 + z^2 = R^2$

Then F (x,y,z) = 8 xyz + $\lambda[x^2 + y^2 + z^2 - R^2]$ and $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0; \frac{\partial F}{\partial z} = 0$ given

$8yz + 2\lambda x = 0$; $8xz + 2\lambda y = 0$; $8xy + 2\lambda z = 0$

$2\lambda x = -8yz$ or $2x^2\lambda = -8xyz = 2y^2\lambda = 2z^2\lambda$

$\Rightarrow 2x^2\lambda = 2y^2\lambda = 2z^2\lambda$

$x^2 = y^2 = z^2 \Rightarrow x = y = z$

$\therefore$ The Rectangular solid in a cube.

Assignment-Cum-Tutorial Questions

A. Questions testing the remembering / understanding level of students

I. I) Objective Questions:

1. If u = xlog(xy) where $x^3 + y^3$+3xy = 1 find $\frac{du}{dx}$.
2. If z = f (x,y), then write $\frac{\partial z}{\partial x}$?
3. If $u = e^{xyz}$, write the values of u_z, u_x,u_y
4. If r = x /y, s = y/z, t = z/x write the value of u_x, u_y, u_z
5. If x = r cosθ, y = r sinθ, find $\frac{\partial r}{\partial x}, \frac{\partial x}{\partial \theta}$.
6. Explain Jacobian?
7. What is the value of J J^1=?
8. Explain extreme value?
9. Write the values of l,m,n value when f (x,y) = 0 in the sense of maximum and minimum?

II. Descriptive Questions

1. If $r^2 = x^2 + y^2 + z^2$ and u = r^m then Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ = m(m+1)r^{m-2}
2. If f (x,y) = $4x^3 - 3x^2 y^2 + 2x + 3y$,Find f_x, f_y, f_{xx}, f_{xy}, f_{yy}
3. If f (x,y) = Tan^{-1} (x + 2y) ,Find f_x, f_y
4. If f (u,v,t) = e^{uv} sin ut, Find f_u, f_v, f_t
5. If z is a function of x and y, where $x^2 + y^2 + z^2 = 1$ Find $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$
6. If f (x,y) = cos 3x X sin 4y find $f_x\left(\frac{\pi}{12}, \frac{\pi}{6}\right)$ and $f_y\left(\frac{\pi}{12}, \frac{\pi}{6}\right)$
7. For f (x,y) = x^7 log y + sin xy, Verify $f_{xy} = f_{yx}$
8. If u = $x^2 - 2y^2 + z^2 + z^3$, x sin , y = e^t, z = 3t find $\frac{du}{dt}$
9. If z = $u^3 v^5$, where u = x + y, v = x- y find $\frac{\partial z}{\partial y}$ by the chain rule.
10. If f (u,v,w) is differentiable, and u = x – y , v = y –z and w = z-x show that $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 6$.

B. Question testing the ability of students in applying the concepts.

I) Multiple Choice Questions

1. Total derivative of u(x, y) is du = [ ]

   a) $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}$ b) $\frac{\partial u}{\partial x}\,.\,dx+\frac{\partial u}{\partial y}\,.\,dy$ c) $\frac{\partial u}{\partial x}\,dx-\frac{\partial u}{\partial y}\,.\,dy$ d) $\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}$

2. $J\,.\,J^1$ = --------- [ ]

   a) 1 b) Zero c) -1 d) none

3. If u = sin (x +y) then $\frac{\partial u}{\partial y}$ =---------- [ ]

   a) sin x b) cos (x + y) c) tan (x + y) d) none

4. If u = $J\left(\frac{u,v}{x,y}\right)$ then $J\left(\frac{x,y}{u,v}\right)$ = [ ]

   a) u b) 1/u c) 1 d) none

5. The minimum value of $x^2 + y^2 + z^2$ given that x+y+z = 3a is [ ]

   a) 3a b) $4a^2$ c) $\frac{a^2}{3}$ d) $3a^2$

6. The stationary points of $x^3 y^2 (1-x-y)$ are [ ]

   a) (0,1) b) (-1,-1) c) (1/2, 1/3) d) (1,1)

7. If the functions u & v of the independent variables x & y are functionally dependent then [ ]

   a) J = 0 b) $J \neq 0$ c) J = 1 d) $J \neq 1$

8. If $ln - m^2 > 0$ & $l < 0$ then f(x, y) has [ ]

   a) minim mum value b) maximum value
   c) zero value d) neither maximum nor minimum

9. If $f(x, y) = x^2 + y^2 + 6x + 12$ then minimum value f(x, y) is [ ]

   a) -3 b) 3 c) 0 d) none

10. If f_x (a, b) = 0 , f_y(a , b) = 0 then (a, b) is said to be [ ]

    a) saddle point b) stationary point c) minimum point d) maximum point

II. Problems:

1. If x+y+z=u, y+z=uv, z=uvw, then evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

2. If u = $x^2\, Tan\frac{y}{x} - y^2 Tan^{-1}\frac{x}{y}$ show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$

3. If $\theta = t^n e^{-r^2/4t}$ what value of n will make $\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)\right] = \frac{\partial\theta}{\partial t}$

4. Given that u = $e^{r\cos\theta}\cos(r\sin\theta)$

   V = $e^{r\cos\theta}\sin(r\sin\theta)$

   Prove tht $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta};\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

5. If f (x,y) = $(1\text{-}2xy + y^2)^{-1/2}$ show that $\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial f}{\partial x}\right]+\frac{\partial}{\partial y}\left[y^2\frac{\partial f}{\partial y}\right]=0$
6. u = f (r); x = r cosθ; y = r sin θ prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r)$
7. If u = $\frac{yz}{x}; v=\frac{xz}{y}, w=\frac{xy}{z}$ show that $\frac{\partial(u,v,w)}{\partial(x,y,z)}=4$
8. Show that the function u = x + y +z , v = $x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$ and w = $x^3 + y^3 + z^3 - 3xyz$ are functionally related?
9. Find the max and min values of $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$?
10. Find the Max and min values of $xy+\frac{e^3}{x}+\frac{e^3}{y}$
11. Find there positive numbers whose sum is 100 and whose product is maximum?
12. Find the min value of $x^2 + y^2 + z^2$ where ax + by + cz = p.
13. A rectangular box open at the top has a capacity of 32 cubic feet. Find the dimensions of the box requiring least material for its construction.

* * *

Linear Algebra & Differential Equations

Unit – VI (Partial Differential Equations)

Course Objectives

- To know how a partial differential equation be formed.
- To know procedure of solving linear first order P.D.E.
- To understand different solution procedures for different types of non-linear P.D.E.s

Syllabus

- Introduction to PDE
- Formation of PDE by elimination of arbitrary Functions
- Solutions of First order Linear equations
- Charpit's method

Learning Outcomes

Students will be able to

- express physical problems in terms of P.D.E
- solve linear first order P.D.E.s
- solve non-linear P.D.E.s

Learning Material

Introduction

The Partial Differential Equation (PDE) corresponding to a physical system can be formed, either by eliminating the arbitrary constants or by eliminating the arbitrary functions from the given relation. The Physical system contains arbitrary constants or arbitrary functions or both.

Equations which contain one or more partial derivatives are called Partial Differential Equations. Therefore, there must be atleast two independent variables and one dependent variable.

Order: The Order of a partial differential equation is the order of the highest partial derivative in the equation.

Degree: The degree of the highest partial derivative in the equation is the Degree of the PDE.

Notation :

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}$$

Formation of P.D.E.

By elimination of arbitrary functions

Form the P.D.E. for the following problems

1) $z = a.\log\left[\frac{b(y-1)}{1-x}\right]$

Sol. $z = a [\log b + \log (y-1) - \log(1-x)]$

Hence $p = a\left[\frac{1}{1-x}\right] \Rightarrow a = p(1-x)$

And $q = a\left[\frac{1}{y-1}\right] \Rightarrow a = q(y-1)$

From these two equations $p(1-x) = q(y-1)$ is the P.D.E.

----

2) x.y.z. = f(x + y + z)

Sol. diff. partially w. r. t 'x' , y[x .p + z] = f^1(x+y+z).(1+p)

diff. partially w. r. t 'y' , x[y .q + z] = f^1(x+y+z).(1+q)

Hence P.D.E is y[x .p + z]/(1+p) = x[y .q + z]/(1+q)

----

3) z = x y + $f(x^2 + y^2)$

Sol. p = y + $f^1(x^2+y^2).(2x)$

q = x + $f^1(x^2+y^2).(2y)$

Hence the P.D.E. is (p-y)/2x = (q –x)/2y

----

4) xy + yz + zx = $f\left[\dfrac{z}{x+y}\right]$

Sol. diff. partially w. r. t 'x' ,

$$y + yp + [z + xp] = f^1\left[\frac{z}{x+y}\right].\left[\frac{(x+y).p - z.1}{(x+y)^2}\right]$$

diff. partially w. r. t 'y',

$$x + [z + yq] + qx = f^1\left[\frac{z}{x+y}\right].\left[\frac{(x+y).q - z.1}{(x+y)^2}\right]$$

Dividing the two equation we get the P.D.E.

<u>SOLUTION OF P.D.E.</u>

<u>FIRST ORDER LINEAR P.D.E.</u>

The general form of a linear P.D.E. of 1^{st} order is P. p + Q. q = R

Where P,Q and R are functions of x, y, and z.

Finding Solution : *<u>LAGRANGE'S METHOD OF SOLUTION</u>*

Lagrange's Auxiliary (Subsidiary) equation is

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}.$$

<u>PROBLEMS</u> : **(USING METHOD OF GROUPING)**

1. Solve p x + q y = z

Sol. It is like P p + Q q =R. Comparing with the above eqn.,

Hence P = x Q = y and R = z.

Hence the Auxiliary equation is $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$ Clearly , it is method of grouping .

$\dfrac{dx}{x} = \dfrac{dy}{y} \Rightarrow$ log x = logy+logc_1 $\Rightarrow$ c_1 = x/y and

$\frac{dy}{y} = \frac{dz}{z} \Rightarrow$ log y = log z + logc_2 $\Rightarrow$ c_2 = y/z

Hence the solution is f(x/y , y/z) = 0

---2. Solve y z p + z x q = x.y

Sol. . It is like P p + Q q =R. Comparing with the above eqn.,

Hence P = y z Q = z x and R = x y.

Hence the Auxiliary equation is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\Rightarrow \frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$ we can separate. So we use method of grouping .

$\frac{dx}{yz} = \frac{dy}{zx} \Rightarrow x.dx = ydy$ integrating $x^2/2 = y^2/2 + c_1$ $\Rightarrow c_1 = x^2/2 - y^2/2$.

$\frac{dy}{zx} = \frac{dz}{xy} \Rightarrow$ y. dy = z.dz integrating $y^2/2 = z^2/2 + c_2$ $\Rightarrow c_2 = y^2/2 – z^2/2$.

Hence the solution is $f(x^2/2 - y^2/2, y^2/2 – z^2/2) = 0$.

----

3. Solve p Tan x + q tan y = z

Sol. . It is like P p + Q q =R. Comparing with the above eqn.,

Hence P = Tan x Q = Tan y and R = z.

Hence the Auxiliary equation is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\Rightarrow \frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{z}$ $\Rightarrow$ Cot x .d x = Cot y . d y = $\frac{dz}{z}$ Clearly we use method of grouping .

Cot x .d x = Cot y . d y integrating log(Sin x) = log(Sin y)+log c_1 $\Rightarrow c_1 = \frac{\sin x}{\sin y}$

Cot y . d y = $\frac{dz}{z}$ integrating log (sin y) = log z + log c_2 $\Rightarrow c_2 = \frac{\sin y}{z}$.

Hence the solution is $f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{z}\right) = 0$.

<u>PROBLEMS</u> : **(USING METHOD OF MULTIPLIERS)**

(WE HAVE 2 DIFFERENT TYPES OF PROCEDURES)

<u>(TYPE-I)</u>

1. Solve x (y – z) p +y (z – x).q = z(x – y)

Sol. It is like P p + Q q =R. Comparing with the above eqn.,

Hence P = x(y-z) Q = y(z-x) and R = z(x-y)

Hence the Auxiliary equation is $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$\Rightarrow \frac{dx}{x(y-z)}=\frac{dy}{y(z-x)}=\frac{dz}{z(x-y)}$ Here we cannot separate x in one side , y in one side and z in one side . So we have to use Method of Multipliers.

Finding Multipliers:

Here P = x(y-z) Q = y(z-x) and R = z(x-y)

Choose i , j and k such that i. P + j. Q + k. R = 0

i.e., i. x(y-z) + j. y(z-x) + k. z(x-y) = 0 ----------→(1)

For i=1, j=1,k=1. Equation (1) satisfies.

Hence one solution is obtained by integrating i. d x +j .d y +k . d z=0

i.e., integrating 1. d x +1.d y + 1 . dz = 0

Solution is x + y + z = c_1.

And For i = 1/x , j = 1/y , k = 1/z also, equation (1) satishies.

Hence one solution is obtained by integrating i. d x +j .d y +k . d z=0

i.e., integrating $\frac{1}{x}$. d x + $\frac{1}{y}$.d y + $\frac{1}{z}$. dz = 0

Solution is log x + log y + log z = log c_2. $\Rightarrow$ c_2 = x.y.z.

Hence solution is f(x+y+z , xyz) = 0

---

2. Solve $x(y^2+z)\,p - y(x^2+z)\,q = z\,(x^2+y^2)$

Sol. It is like P p + Q q =R. Comparing with the above eqn.,

Hence P = $x(y^2$ +-z) Q = - $y(x^2 + z)$ and R = $z(x^2+y^2)$

Hence the Auxiliary equation is $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$\Rightarrow \frac{dx}{x(y^2+z)}=-\frac{dy}{y(x^2+z)}=\frac{dz}{z(x^2+y^2)}$ Here we cannot separate x in one side , y in one side and z in one side . So we have to use Method of Multipliers.

Finding Multipliers: Here P = $x(y^2$ +-z) Q = - $y(x^2 + z)$ and R = $z(x^2+y^2)$

Choose i , j and k such that i. P + j. Q + k. R = 0

i.e., i. $x(y^2$ +-z) + j.(- $y(x^2 + z)$) + k. $z(x^2+y^2)$ = 0 ---------→(1)

For i = 1/x , j = 1/y and k = 1/z Equation (1) satisfies.

Hence one solution is obtained by integrating i. d x +j .d y +k . d z=0

i.e., integrating $\frac{1}{x}$. d x + $\frac{1}{y}$.d y + $\frac{1}{z}$. dz = 0

Solution is log x + log y + log z = log c_1. $\Rightarrow$ c_1 = x.y.z. //

For i = x , j = y and k = -1 also Equation (1) satisfies.

Hence one solution is obtained by integrating i. d x +j .d y +k . d z=0

i.e., integrating x. d x +y.d y + (-1) . dz = 0 we get $x^2/2 + y^2/2 - z = c_2$.

Hence Solution is $f(xyz , x^2/2 + y^2/2 - z)=0$.

(TYPE-II)

1. Solve (y+z)p+(z+x)q=x+y

Sol. It is like P p + Q q =R. Comparing with the above eqn.,

Hence P = (y+z) Q = (z+x) and R = (x+y)

Hence the Auxiliary equation is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\Rightarrow \frac{dx}{(y+z)} = \frac{dy}{(z+x)} = \frac{dz}{(x+y)}$ Here we cannot separate x in one side , y in one side and z in one side . So we have to use Method of Multipliers.

Finding Multipliers:

Here P = (y+z) Q = (z+x) and R = (x+y)

Choose i , j and k such that i. P + j. Q + k. R = 0

i.e., i.(y + z) + j.(z +x) + k.(x +y) = 0 --------→(1)

<u>For this we cannot find multipliers by observation.</u>

Write $\frac{dx}{(y+z)} = \frac{dy}{(z+x)} = \frac{dz}{(x+y)}$ as $\frac{i.dx + j.dy + k.dy}{i(y+z) + j(z+x) + k(x+y)}$ --→(a)

So Choosing 1,-1,0 as multipliers, (a) becomes $\frac{dx - dy}{y + z + (-1)(z + x)} = \frac{d(x-y)}{-(x-y)}$

Choosing 0,1,-1 as multipliers, (a) becomes $\frac{dy - dz}{z + x + (-1)(x + y)} = \frac{d(y-z)}{-(y-z)}$

Choosing 1,1,1 as multipliers , (a) becomes $\frac{dx + dy + dy}{(y+z) + (z+x) + (x+y)} = \frac{d(x+y+z)}{2(x+y+z)}$

$$\frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} = \frac{d(x+y+z)}{2(x+y+z)}$$

From 1st two fractions ,integrating log(x-y) -log(y-z)=log $c_1 \Rightarrow c_1 = \frac{x-y}{y-z}$ //

From 1st and 3rd , integrating we get $c_2 = (x-y)^2.(x+y+z)$ //

Hence the solution is $f(\frac{x-y}{y-z}, (x-y)^2.(x+y+z)) = 0$.

---

2. Solve $(x^2 - yz)\, p + (y^2 - zx)\, q = z^2 - xy$

Sol. It is like P p + Q q =R. Comparing with the above eqn.,

Hence $P = (x^2 - yz)$ $Q = (y^2 - zx)$ and $R = (z^2 - xy)$

Hence the Auxiliary equation is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\Rightarrow \frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$ Here we cannot separate x in one side , y in one side and z in one side . So we have to use Method of Multipliers.

Finding Multipliers:

Here $P = (x^2 - yz)$ $Q = (y^2 - zx)$ and $R = (z^2 - xy)$

Choose i , j and k such that i. P + j. Q + k. R = 0

i.e., $i.(x^2 - yz) + j.(y^2 - zx) + k.(z^2 - xy) = 0$ ----------→(1)

<u>For this we cannot find multipliers by observation.</u>

Write $\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$ as $\frac{i.dx + j.dy + k.dy}{i(x^2 - yz) + j(y^2 - zx) + k(z^2 - xy)}$ --→(a)

So Choosing 1,-1,0 as multipliers, (a) becomes $\frac{dx - dy}{x2 - yz - y2 + zx} = \frac{d(x-y)}{(x^2 - y^2) + z(x-y)}$

Choosing 0,1,-1 as multipliers, (a) becomes $\frac{dy - dz}{y^2 - zx - z^2 + xy} = \frac{d(y-z)}{y^2 - z^2 + x(y-z)}$

Choosing 1,1,1 as multipliers , (a) becomes $\frac{dx + dy + dy}{(x^2 - yz) + (y^2 - zx) + (z^2 - xy)}$

Hence $\frac{d(x-y)}{(x^2 - y^2) + z(x-y)} = \frac{d(y-z)}{y^2 - z^2 + x(y-z)} = \frac{dx + dy + dy}{(x^2 - yz) + (y^2 - zx) + (z^2 - xy)}$

From this also we don't get a solution. Hence take another multiplier as x,y,z

We get $\frac{xdx + ydy + zdy}{x(x^2 - yz) + y(y^2 - zx) + z(z^2 - xy)} = \frac{xdx + ydy + zdy}{x^3 + y^3 + z^3 - 3xyz}$

$$= \frac{xdx + ydy + zdy}{(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)}$$

Hence all the four fractions are equal.

So

$$\frac{d(x-y)}{(x^2 - y^2) + z(x-y)} = \frac{d(y-z)}{y^2 - z^2 + x(y-z)} = \frac{dx + dy + dy}{(x^2 - yz) + (y^2 - zx) + (z^2 - xy)} = \frac{xdx + ydy + zdy}{(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)}$$

From 1st two fractions,

$\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(x+y+z)}$ Integrating $c_1 = \frac{x-y}{y-z}$.

From last two fractions. $\frac{dx + dy + dz}{1} = \frac{xdx + ydy + zdz}{x + y + z}$ Integrating

$C_2 = xy + yz + zx$

Hence the solution is $f(\frac{x-y}{y-z}, xy + yz + zx) = 0$.

----

3. Solve $y^2(x-y)\,p + x^2\,(y-x)q = z(x^2+y^2)$

Sol. It is like $P\,p + Q\,q = R$. Comparing with the above eqn.,

Hence $P = y^2(x-y)$ $Q = x^2(y-x)$ and $R = z(x^2+y^2)$.

Hence the Auxiliary equation is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\Rightarrow \frac{dx}{y^2(x-y)} = \frac{dy}{-x^2(x-y)} = \frac{dz}{z(x^2+y^2)}$$

From 1st two fractions, we get one solution by grouping method.

$x^2.dx = -y^2dy$ Hence $c_1 = x^3 + y^3$.

To get another solution, we cannot separate x in one side , y in one side and z in one side . So we have to use Method of Multipliers.

Finding Multipliers:

Here $P = y^2(x-y)$ $Q = x^2(y-x)$ and $R = z(x^2+y^2)$.

Choose i , j and k such that $i.\,P + j.\,Q + k.\,R = 0$

i.e., $i.\ y^2(x-y) + j.\ x^2(y-x) + k\,.\,z(x^2+y^2) = 0$ --------→(1)

<u>For this we cannot find multipliers by observation.</u>

Write $\frac{dx}{y^2(x-y)} = \frac{dy}{-x^2(x-y)} = \frac{dz}{z(x^2+y^2)}$ as $\frac{i.dx + j.dy + k.dy}{iy^2(x-y) - jx^2(x-y) + kz(x^2+y^2)}$

--→(a)

So Choosing 1,-1,0 as multipliers, (a) becomes

$$\frac{dx - dy}{y^2(x-y) + x^2(x-y)} = \frac{d(x-y)}{(x-y)(x^2+y^2)}$$

Clearly $\frac{d(x-y)}{(x-y)(x^2+y^2)} = \frac{dz}{z(x^2+y^2)}$ Integrating $c_2 = (x-y)/2$.

Hence the solution is $f(\ x^3 + y^3, (x-y)/2)$.

NON-LINEAR P.D.E.

A partial differential equation which involves first order partial derivatives and with degree higher than one and the products of and is called a non-linear partial differential equation.

Fallowing are the types of non-linear partial differential equations of first order :

Type I: $f(p, q)=0$

Type II: $f(p, q, z)=0$

Type III: $f(p, x) = g(q, y)$ (variable separable method)

Type IV: Clairaut's form

Type 1:

<u>When the problem is a function of p and q , i.e., f(p, q)=0</u>

Procedure : The complete integral (solution) is given by $z = a\,x + b\,y + c$

Where $\mathbf{p} = \frac{\partial z}{\partial x} = \mathbf{a}$ **and** $\mathbf{q} = \frac{\partial z}{\partial y} = \mathbf{b}$

Problems :

1. Find the Complete integral of $p^3 - q^3 = 0$

Sol. Clearly given is f(p,q)=0

Hence the Complete integral is z= a x + b y + c

Where $\mathbf{p} = \frac{\partial z}{\partial x} = \mathbf{a}$ **and** $\mathbf{q} = \frac{\partial z}{\partial y} = \mathbf{b}$

Substituting p and q values in given we get $a^3 - b^3 = 0 \Rightarrow a = b$

Hence the solution is z = ax + ay + c

----

2. Solve p q = k

Sol. Clearly given is f(p, q) = 0.

Hence the Complete integral is z= a x + b y + c

Where $\mathbf{p} = \frac{\partial z}{\partial x} = \mathbf{a}$ **and** $\mathbf{q} = \frac{\partial z}{\partial y} = \mathbf{b}$

Substituting p and q values in given we get a. b = k $\Rightarrow$ b=k/a

Hence the solution is z = ax + ky/a + c

----

3. Find the complete solution of p + q = p. q & hence find the general solution.

Sol. It is clearly f(p, q) = 0.

Hence the Complete integral is z= a x + b y + c

Where $\mathbf{p} = \frac{\partial z}{\partial x} = \mathbf{a}$ **and** $\mathbf{q} = \frac{\partial z}{\partial y} = \mathbf{b}$

Substituting p and q values in given we get a + b = a. b $\Rightarrow$ b = a/(a-1)

Hence the complete solution is z = a. x + a. y /(a-1) + c.

For finding general solution , Take c = f(a)

$$\Rightarrow z = a.\ x + a.\ y/(a-1) + f(a) \quad \text{------------}\rightarrow(1)$$

Diff. w.r.t 'a' we get $0 = x - \frac{y}{(a-1)^2} + f^1(a)$ ------------→(2)

Eliminating 'a' from (1) 7 (2) we get the solution.

Type 2 :

EQUATIONS INVOLVING Z , p and q. i.e., f(z,p,q)=0.

Procedure :

1. Substitute q = a p in the given equation and find p .
2. then we can find q from p
3. Integrating the equation d z = p d x + q d y we get the soln.

Problems:

1. Solve z p q = p + q.

Sol. Substitute q = a p , we get $z\,a\,p^2 = p + a.\,p$

$$\Rightarrow p = \frac{1+a}{az} \quad \text{and} \quad q = \frac{1+a}{z}$$

Hence Integrating the equation $dz = \frac{1+a}{az}.dx + \frac{1+a}{z}.dy$ we get

$az^2/2 = (1+a).[x + ay] + k.$

----

2. Solve $p(1 + q) = qz$

Sol. . Substitute $q = ap$, we get $p(1 + ap) = apz \Rightarrow p = \frac{az-1}{a}$

Hence $q = az - 1$.

Hence integrating the equation $dz = \frac{az-1}{a} dx + (az-1).dy$ we get

$\Rightarrow dz = \frac{az-1}{a} [dx + a.dy] =$ //

----

3. Solve $p^2 z^2 + q^2 = p^2.q$

Sol. Substitute $q = ap$, we get $p^2 z^2 + a^2.p^2 = p^2.ap \Rightarrow p = \frac{z^2+a^2}{a^2}$

And hence $q = \frac{z^2+a^2}{a}$

Hence integrating the equation $dz = \frac{z^2+a^2}{a^2} dx + \frac{z^2+a^2}{a} dy$ we get

$\frac{a^2 dz}{z^2+a^2} = dx + a.dy \Rightarrow \text{Tan}^{-1}(z/a) = x + a.y + k$

Type 3 : $f(x, p) = g(y, q)$

Problems:

1. solve $p^2 + q^2 = x + y$

Sol. We can write $p^2 - x = y - q^2 = a$ (Assume)

Hence $p = \sqrt{a+x}$ and $q = \sqrt{y-a}$

Substitute in $dz = p.dx + q.dy$ we get

$dz = \sqrt{a+x}.dx + \sqrt{y-a}.dy$ integrating

$z = 2(a+x)^{3/2}/3 + 2(y-a)^{3/2}/3 + k$

----

2. solve $p - q = x^2 + y^2$

Sol. $p - x^2 = q + y^2 = a$ (Assume)

Then $p = a+x^2$ and $q = a - y^2$

Substitute in $dz = p.dx + q.dy$ we get

$dz = (a + x^2).dx + (a - y^2).dy$ integrating

$z = ax + x^3/3 + ay - y^3/3 + k$

----

3. Solve $yp + xq + pq = 0$

Sol. This can be written as $(x + p).q = -yp \Rightarrow \frac{x+p}{p} = -\frac{y}{q} = a$

Continue.

Type 4 : <u>CLAIRAUT'S FORM :</u>

Procedure: The form $z = px + qy + f(p, q)$ is known as Clairaut's form.

The solution is $z = ax + by + f(a,b)$

Problems:

1. Solve $z = px + qy + pq$

Sol. Clearly it is in Clairaut's form Hence solution is $z = ax + by + ab$.

2.solve $z = px + qy + \sqrt{p^2 + q^2 + 1}$

Sol. Clearly it is in Clairaut's form Hence solution is $z = ax + by + \sqrt{a^2 + b^2 + 1}$

----

3. Solve $z = px + qy - n\, p^{1/n}.q^{1/n}$

Sol. Clearly it is in Clairaut's form Hence solution is

$$z = ax + by - n.a^{1/n}.b^{1/n}.$$

Type 5. CHARPIT'S METHOD (General method)

For Given $f(x, y, z, p, q) = 0$

Take the equation $\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$

And then solve.

1.Solve $p^2 - y^2.q = y^2 - x^2$

Sol. Write the given function as $f(x,y,z,p,q) = p^2 - y^2.q - y^2 + x^2$ -----→(1)

Substitute in the above equation , we get

$$\frac{dx}{-2p} = \frac{dy}{y^2} = \frac{dz}{-p(2p) - q(-q^2)} = \frac{dp}{2x} = \frac{dq}{-2qy - 2y}$$

From 1st and 4th fraction, we get $p^2 + x^2 = a^2$. ------------------→(3)

Solving (1) & (3) $p = \sqrt{(a^2 - x^2)}$ and $q = \frac{a^2}{y^2} - 1$

Substituting these in $dz = p.dx + q.dy$ and integrating, we get the solution.

----

2. Solve $z^2 = pqxy$ by charpit's method.

Sol. $f(x, y, z, p, q) = z^2 - pqxy = 0.$ --------------------------→(1)

Substituting in Charpit's equation, we get

$$\frac{dx}{-qxy} = \frac{dy}{-pxy} = \frac{dz}{-2pqxy} = \frac{dp}{-pqy + p.2z} = \frac{dq}{-pqx - 2qz}$$

$$\frac{p.dx + q.dy}{-2pxz} = \frac{q.dy + y.dq}{-2qyz} \quad \Rightarrow \quad \frac{d(px)}{px} = \frac{d(qy)}{qy}$$ Integrating,

we get p x = k. q. y --→(2)

Solving (1) and (2) we get $p = \frac{k.z}{x}$ and $q = \frac{z}{k.y}$

Substituting in dz = p d x + q d y and integrating we get $z = c\ x^k . y^{1/k}$.

Assignment-cum-Tutorial Questions

A)Questions testing the remembering / understanding level of students.

I)Objective Questions

1. The general solution of z = px+qy+p/q is______________
2. The general solution of $p^2 + q^2 = m^2$ is __________
3. The Complete integral of $f(p,q) = 0$ is ___________
4. The Complete integral of $p^2q^3 = 1$ _______________
5. If the number of arbitrary constants to be eliminated is equal to the number of independent variables then we get a partial differential equation of _________order
6. If the number of arbitrary constants to be eliminated is greater than the number of independent variables then we get a partial differential equation of _________ order
7. The partial differential equation of all spheres whose centres lie on the z-axis is_______
8. Lagrange's subsidiary equation is ___________
9. The general solution of $\sqrt{p} + \sqrt{q} = 1$ is _________
10. The Complete integral of $p^2q^3 = 1$ _______________
11. The general solution of $pq + p + q = 0$ is __________
12. The general solution of $p^2 - q^2 = 4$ is ___________
13. General form of Clairauti equation is__________
14. The general solution of $z = px + qy + f(p,q)$ is ___________
15. The general solution of $z = px + qy + \log pq$ is ____________
16. The general solution of $z = px + qy + \sqrt{1 + p^2 + q^2}$ is __________

B. Questions testing the ability of students in applying the concepts

II) Objective Questions

1. The general solution of $p^2 + q^2 = x + y$ is ______
2. The partial differential equation by eliminating the arbitrary constants from $z = (x^2 + a)(y^2 + b)$ is___ ___
3. The partial differential equation by eliminating arbitrary function from $z = f(x^2 + y^2)$ is____

4. The partial differential equation by eliminating arbitrary function from $z = x^n f(y/x)$ is _____

5. The partial differential equation by eliminating arbitrary function from $z = y f(y/x)$ is______

6. The partial differential equation by eliminating the arbitrary function from the relation $z = f(\sin x + \cos y)$ is __________

7. The partial differential equation by eliminating the arbitrary function from the relation $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ is ________

8. The general solution of $dx = dy = dz$ is _________

9. The general solution of $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$ is _________

10. Lagrange's equation $Pp + Qq = R$ is non-linear partial differential equation [True/ False]

11. The set of multiplies to solve the equation $(x^2 - yz)p + (y^2 - zx) = z^2 - xy$ is [ ]

    a) $-1,1,0$ b) $0,-1,1$ c) $both\ a\ \&\ b$ d) $1,-1,0$

12. The Lagrange's subsidiary equation where P, Q, R are functions of x, y and z is [ ]

    a) $Pdx + Qdy + Rdz = 0$ b) $dx + dydz = 0$ c) $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ d) None

13. The partial differential equation by eliminating arbitrary function from $z = f(x^2 - y^2)$ is [ ]

    a) $px - qy = 0$ b) $py + qx = 0$ c) $p(x^2) - q(y^2) = 0$ d) None

Problems:

1. Form the partial differential equations from the following relations:

   (i) $xyz = f(x + y + z)$ (ii) $z = f(x + at) + g(x - at)$

   (iii) $f(x^2 + y^2, z - xy) = 0$ (vi) $f(x + z, y + z) = 0$ (v) $z = y f(x) + x g(y)$

2. Solve (i) $pq + p + q = 0$ (ii) $(p + q)(z - px - qy) = 1$.
3. Solve $(2z - y)p + (x + z)q = -(2x + y)$
4. $Solve\,(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$.
5. Solve $p \tan x + q \tan y = \tan z$.
6. $Solve\,(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$
7. Solve the partial differential equation (y + z) p + (z + x) q = x + y.
8. Solve x (y – z) p + y (z –x) q = z (x-y).
9. Solve $px^2 - qy^2 = z(x - y)$
10. Solve $(mz - ny)p + (nx - lz)q = lz - mx$
11. Solve $z^2 = pqxy$.
12. Solve $z = p^2x + q^2y$.
13. Solve $pxy + pq + qy = yz$ by using Charpit's method.
14. Solve $z(z^2 + xy)(px - qy) = x^4$
15. Solve $p + 3q = 5z + \tan(y - 3x)$
16. Solve $(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x - y)$.
17. Find the integral surface of $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ which contains the straight line $x + y = 0, z = 1$.
18. Find the general solution of $y^2zp + x^2zq = y^2x$

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